Exam Complex Function Theory
Code 201500405

Date	:	Friday July 9, 2021
Place	:	CR-2K
Time	:	09.00 – 11.00

All answers must be motivated.

The use of a pocket calculator or any other electronic equipment is not allowed.

At the end of the exam, just before you hand in your work, please scan or photograph your work, and upload these (pdf) files on Canvas, before 8 o'clock this evening.

1. Let f be an analytic function in the domain D. Assume that f takes all its values on the line $\ell = \{w \in \mathbb{C} \mid w = a + ia, a \in \mathbb{R}\}$, i.e., $f(z) \in \ell$ for all $z \in D$.

   Show that f is constant.

2. Let α and β be complex numbers. Show that for $z \neq 0$ the following holds,

   $$z^{\alpha+\beta} = z^{\alpha} z^{\beta},$$

   where each power function is given by its principal branch.

3. Define the function $h(z)$ by

   $$h(z) = \frac{1}{\sin(z)} - \frac{1}{z}. \tag{1}$$

   (a) Show that h is meromorphic in $\mathbb{C}$.

   (b) Show that $z = 0$ is a removable singularity.

   (c) Find and characterise all singularities of h inside the disk $\mathbb{D} = \{z \in \mathbb{C} \mid |z| < 2\pi\}$.

   (d) Classify the behaviour at ∞ for $h(z)$.

4. Given is the entire function $f(z)$ satisfying

   $$f(x) = x^3 - 3x + 5, \qquad x \in \mathbb{R} \tag{2}$$

   Construct $f(z)$. Is it unique?

P.T.O.

5. Determine the outcome of the following integral

$$\int_{\Gamma} \frac{z-3}{z(z^2+1)}dz,$$

where Γ is the contour drawn in Figure 1

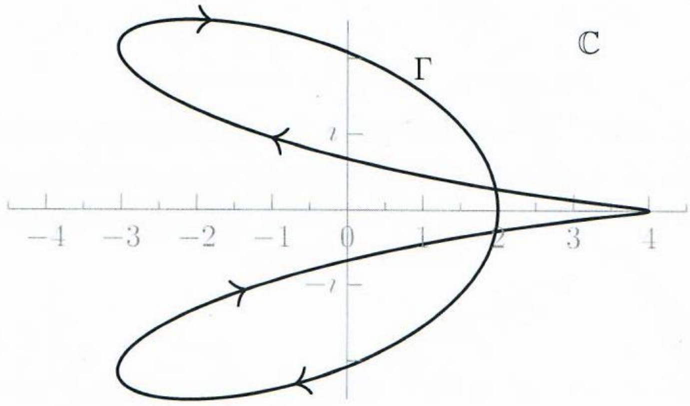

Figure 1: The curve Γ.

6. Let g be function analytic in a domain D containing $\Omega = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$.

   Can g have infinitely many zeros within Ω?

   If no, prove why. If yes, give an example of such a g.

7. Determine the integral

$$\text{P.V.} \int_{-\infty}^{\infty} \frac{\sin(x)}{x}dx,$$

   where P.V. stands for principle value.

8. Suppose that $f(z)$ is analytic on a domain containing the closed unit disk, $\{z \in \mathbb{C} \mid |z| \leq 1\}$, and satisfies $|f(z)| < 1$ for $|z| = 1$.

   Show that the equation $f(z) = z^2$ has exactly two (counting multiplicity) solutions inside the open unit disk $\{z \in \mathbb{C} \mid |z| < 1\}$.

Points[1]

Ex. 1	Ex. 2	Ex. 3		Ex. 4	Ex. 5	Ex. 6	Ex. 7	Ex. 8
3	4	a	2	3	4	4	4	4
		b	2					
		c	3					
		d	3					

[1] Total: $36 + 4 = 40$ points