Exam Complex Function Theory
Code 201500405

Date	:	Wednesday June 8, 2022
Place	:	NH-115
Time	:	08.45 – 10.45

All answers must be motivated.

The use of a pocket calculator or any other electronic equipment is not allowed.

1. On the non-empty domain Ω the function $f(z)$ is defined. Prove that if f and $\overline{f}$ are analytic on Ω, then f is constant.

2. Of each of the following statements determine if it is true or not. If it holds, provide a proof, otherwise show that it does not hold, for instance via a counterexample.

   (a) The function $u(x, y) = x^2 + y^2$ is harmonic.

   (b) The integral of $f(z) = \frac{1}{z}$ over the half-circle running from $-i$ to i is the same when taking the route via γ_1 or via γ_2, see Figure 1.

   (c) The equality $\sqrt{z^2} = z$ holds for all $z \in \mathbb{C}$.

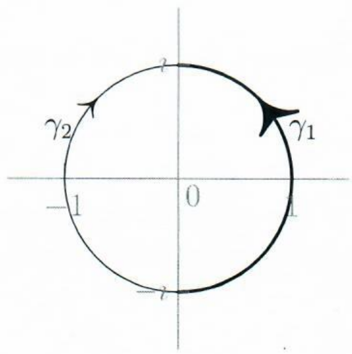

Figure 1: Going left or right, the curves γ_1 and γ_2 going from $-i$ to i.

3. Define the function $h(z)$ by

$$h(z) = \frac{\cos(z) - 1}{z^2} + \frac{1}{z - 1}. \qquad (1)$$

   (a) Show that h is meromorphic in $\mathbb{C}$.

   (b) Show that $z = 0$ is a removable singularity.

   (c) Determine the poles of h and their order.

   (d) Classify the behaviour at ∞ for h.

P.T.O.

4. Determine the integral of $\frac{1}{z(z+1)}$ along the closed curve $\gamma = \{\frac{2e^{2it}}{e^{it}-2} \mid t \in [0, 2\pi]\}$, see Figure 2 for its graph and orientation.

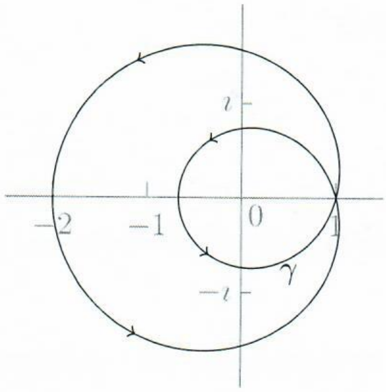

Figure 2: The curve γ.

5. Show that $f(z)$ defined via the power series

$$f(z) = \sum_{k=0}^{\infty} \frac{1}{(k!)^2} z^k, \qquad z \in \mathbb{C} \tag{2}$$

is an entire function.

6. Let $P(z)$ be a polynomial of degree n. For $R > 0$ we consider the integral

$$I(R) := \oint_{|z|=R} \frac{P'(z)}{P(z)} dz.$$

   (a) Prove that the outcome of the integral is non-decreasing with R.
   (b) Show that $\lim_{R\to\infty} I(R)$ exists and determine its value.

7. Determine the integral

$$\text{p.v.} \int_0^{\infty} \frac{1}{x^2-1} dx,$$

where p.v. stands for principal value.

8. It is known that the Laplace transform of

$$h(t) = \begin{cases} e^t, & t > 0 \\ 0 & t < 0 \end{cases}$$

is given by

$$H(s) = \frac{1}{s-1} \qquad [\text{Re}(s) > 1].$$

Use the formula of the inverse Laplace transform (Bromwich integral) to show that the inverse Laplace transform of $H(s)$ equals $h(t)$ for $t \neq 0$.

Points[1]

Ex. 1	Ex. 2		Ex. 3		Ex. 4	Ex. 5	Ex. 6		Ex. 7	Ex. 8
3	a	2	a	2	4	3	a	3	4	4
	b	2	b	2			b	1		
	c	2	c	2						
			d	2						

[1] Total: $36 + 4 = 40$ points