Exam Complex Function Theory
Code 201500405

Date	:	Monday July 11, 2022
Place	:	Therm-1
Time	:	08.45 – 10.45

All answers must be motivated.

The use of the book, or lecture notes, summaries, etc. is not allowed.
The use of a pocket calculator or any other electronic equipment is not allowed.

1. (a) For which $\alpha, \beta \in \mathbb{R}$ is the function

$$u(x, y) = x^2 + \alpha y^2 + \beta xy$$

harmonic?

(b) For the α's and β's found in the previous part, determine the entire function $f(z)$ such that $f(0) = i$ and $u(x, y) = \mathrm{Re}(f(x + iy))$.

2. Of each of the following statements determine if it is true or not. If it holds, provide a proof, otherwise show that it does not hold, for instance via a counterexample.

(a) $\mathrm{Log}(z_1 z_2) = \mathrm{Log}(z_1) + \mathrm{Log}(z_2)$ for all $z_1, z_2 \in \mathbb{C} \setminus \{0\}$.

(b) For $R > 0$, the integral $\oint_{|z|=R} \frac{1}{z} dz$ is independent of R.

(c) There does not exist a non-zero function which is analytic on $D = \{z \in \mathbb{C} \mid \mathrm{Re}(z) > 0\}$ and has infinitely many zeros in D.

3. Define the function $h(z)$ by

$$h(z) = \frac{e^{2z} - 1}{z^2}. \tag{1}$$

(a) Determine the Laurent series of h.

(b) Determine of h the poles with their order.

(c) Classify the behaviour at ∞ for h.

4. Determine the following extremum

$$\max_{|z| \leq 1} \left| \frac{z^2}{z - 2} \right|.$$

P.T.O.

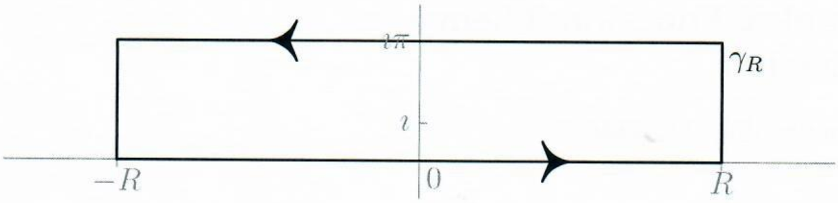

Figure 1: The curve γ_R.

5. Consider the function
$$f(z) = \frac{e^z}{\cosh(2z)}.$$
   (a) Show that $f(z + \pi i) = -f(z)$.
   Hint: $\cosh(z) = \frac{e^z + e^{-z}}{2}$.
   (b) Show that $z_k = \left(\frac{k\pi}{2} + \frac{\pi}{4}\right) i$, $k \in \mathbb{Z}$, are poles of $f(z)$, and determine their order.
   (c) Determine the integral of $f(z)$ along the closed curve γ_R, see Figure 1 for its graph and orientation.
   Hint: You may assume that z_k of item (b) are the only poles of f.
   (d) Determine the following integral
$$\text{p.v.} \int_{-\infty}^{\infty} f(x)dx, \qquad \text{where p.v. stands for principle value.}$$

6. It is easy to show that $g(x) = \sin(x)$ maps the open interval $(-2\pi, 2\pi)$ into the closed interval $[-1, 1]$. We study in this exercise if this can happen for a domain $D \subset \mathbb{C}$.
   Let $f(z)$ be a non-constant entire function, and let $z_0 \in \mathbb{C}$. We define $c \in \mathbb{C}$ as $f(z_0)$, i.e., $f(z_0) = c$.
   (a) Prove that there exists an $r_0 > 0$ such that $f(w) \neq c$ for all $w \in \{z \in \mathbb{C} \mid 0 < |z - z_0| < r_0\}$.
   (b) For $r \in (0, r_0)$ we define $\mathbb{D}_r = \{z \in \mathbb{C} \mid |z - z_0| < r\}$, and
$$\sigma(r) = \min_{|z - z_0| = r} |f(z) - c|.$$
   Show that there exists a $z \in \mathbb{D}_r$ such that
$$f(z) = c + \frac{1}{2}\sigma(r).$$
   (c) Does there exists a domain D such that the sinus function maps D onto $[-1, 1]$?

Points[1]

Ex. 1		Ex. 2		Ex. 3		Ex. 4	Ex. 5		Ex. 6	
a	2	a	2	a	2	3	a	1	a	3
b	3	b	2	b	2		b	2	b	3
		c	2	c	2		c	3	c	1
							d	3		

[1] Total: $36 + 4 = 40$ points