Exam Complex Function Theory
Code 201500405

Date	:	Wednesday June 7, 2023
Place	:	TL-3330
Time	:	08.45 – 10.45

All answers must be motivated.

The use of the book, or lecture notes, summaries, etc. is not allowed.
The use of a pocket calculator or any other electronic equipment is not allowed.

1. (a) Let f be an entire functiom, show that

$$g(z) = f(z)^2$$

is entire as well.

   (b) Is the converse of the above statement true? Hence if $g(z) = f(z)^2$ is entire, does that imply that f is entire?

2. Of each of the following statements determine if it is true or not. If it holds, provide a proof, otherwise show that it does not hold, for instance via a counterexample.

   (a) If $u(x, y)$ and $v(x, y)$ are harmonic, then $f(z)$ defined as $f(x+iy) = u(x, y)+iv(x, y)$ is analytic.

   (b) For all $z \in \mathbb{C}$ there holds that $\mathrm{Log}(z^2) = 2\,\mathrm{Log}(z)$.

   (c) The equation $\sin(z) = i$ has one solution in $\mathbb{C}$.

3. Let f be an analytic function in the disc $D = \{z \in \mathbb{C} \mid |z| < 1\}$. Assume that its derivative is bounded, i.e., $|f'(z)| \leq M$ for all $z \in D$. Prove that

$$|f(z_1) - f(z_2)| \leq M|z_1 - z_2|, \qquad \text{for all } z_1, z_2 \in D.$$

4. Define the function $h(z)$ by

$$h(z) = \frac{e^{z^2} - 1}{z^3}. \tag{1}$$

   (a) Determine the Laurent series of h.

   (b) Determine of h the poles with their order.

   (c) Classify the behaviour at ∞ for h.

P.T.O.

5. Use two different methods/theorems to determine the integral

$$\int_C \frac{z}{4z^2+1}\,dz.$$

Here C is the unit circle, positively oriented.

6. A rational function f is said to be a *positive real function* if $\operatorname{Re}(f(z)) > 0$ whenever the real part of z is positive.

   Show that the positive real rational function f has no zeros with real part strictly larger than zero.

7. Let f be a function satisfying the following properties.

   - f is analytic on $\Omega \supset \{z \in \mathbb{C} \mid \operatorname{Im}(z) \geq 0\}$;
   - for $z \in \mathbb{R}$, $f(z)$ is real valued;
   - There exists an $M > 0$ such that $|f(z)| \leq M/|z|$ for $z \in \Omega$ with $|z| > 1$.

   (a) Prove the following equality

   $$\text{p.v.} \int_{-\infty}^{\infty} \frac{f(x)}{x-i}\,dx = 2\pi i f(i), \qquad \text{where p.v. stands for principle value.}$$

   (b) Next we write f in its real and imaginary part, i.e., $f(x+iy) = u(x,y) + iv(x,y)$. Prove that

   $$\text{p.v.} \int_{-\infty}^{\infty} \frac{xf(x)}{2\pi(x^2+1)}\,dx = -v(0,1).$$

Points[1]

Ex. 1		Ex. 2		Ex. 3	Ex. 4		Ex. 5	Ex. 6	Ex. 7	
a	2	a	2	4	a	3	4	4	a	3
b	3	b	2		b	2			b	2
		c	2		c	3				

[1] Total: $36 + 4 = 40$ points