Exam Graph Theory (191520751)

University of Twente

April 22, 2022, 13:45–16:45

This exam has 6 exercises.
Motivate/justify all your answers by explaining how you got to them.
You may not use electronic devices or lecture materials, books, notes, et cetera.
Don't forget to turn off your cell phone!

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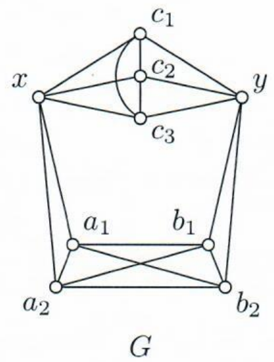
1. Let G be a simple connected graph, and recall that $\tau(G)$ denotes the number of spanning trees of G. Suppose that G contains a vertex v with $d(v) = 2$ and such that $G - v$ is connected.
   (a) Prove that $\tau(G) \geq 2\tau(G - v) + 1$.
   (b) Draw an example of a graph with the above properties for which equality holds in the above inequality. Explain why your example is correct.
   (c) Draw an example of a graph with the above properties for which strict inequality holds in the above inequality. Explain why your example is correct.

2. Let G be a simple connected graph with $\nu(G) \geq 3$ and $\kappa(G) = \Delta(G)$.
   (a) Prove that G is regular.
   (b) Prove: if G is not 3-edge-connected, then G contains a Hamilton cycle.

3. Let G be a simple k-regular graph with $k \geq 2$, and assume that G has a cut edge e.
   (a) Prove that either every perfect matching of G contains e or no perfect matching of G contains e.
   (b) Show that $\chi'(G) = k + 1$.

4. Let G be the graph depicted on the right.
   (a) Determine $\chi(G)$.
   (b) Is G a critical graph? Explain why it is (not).



G_0 G_1

Figure 1: The graphs G_0 and G_1 of exam question 5

5. Let $\pi_k(G)$ denote the chromatic polynomial of a simple graph G. Suppose G contains two adjacent vertices u and v. Let G_w denote the graph obtained from G by adding a new vertex w and the edges uw and vw.

   (a) Prove that $\pi_k(G_w) = (k-2)\pi_k(G)$.

   (b) Show that $\pi_k(G_0) = k(k-1)^3(k-2)^6$ for the graph G_0 depicted in Figure 1.

   (c) Derive an expression for $\pi_k(G_1)$ of the graph G_1 depicted in Figure 1.

6. Recall that the girth of a graph is the length of a shortest cycle in the graph.

   (a) Prove that if G is a connected 4-regular graph with girth at least 4, then G is not planar.

   (b) Give an example of a connected 4-regular graph that is planar.

Norm:

Exercise	1			2		3		4		5			6	
	a	b	c	a	b	a	b	a	b	a	b	c	a	b
Points	3	2	2	2	3	3	4	2	3	3	2	2	3	2

Total: 36 points. Your grade is $\frac{1}{4} \cdot$ (your total score of points plus 4), rounded.