Exam Graph Theory (191520751)

University of Twente

July 15, 2022, 08:45-11:45

This exam has 6 exercises.
Motivate/justify all your answers by explaining how you got to them.
You may not use any electronic device or lecture materials, books, notes, et cetera.
Don't forget to turn off your cell phone!

1. Let T be a tree with $\nu \geq 2$ and $\Delta \leq 3$, and let n_i denote the number of vertices with degree i for $i = 1, 2, 3$.

   (a) Prove that $n_1 = n_3 + 2$.

   (b) Determine the maximum possible number of leaves of T, as a function of ν.

2. For a simple graph G and an edge $e = uv \in E(G)$, let $G \cdot e$ denote the simple graph obtained from $G - \{u, v\}$ by adding a new vertex w and joining w by an edge to each vertex $x \in (N_G(u) \cup N_G(v)) \setminus \{u, v\}$ (where $N_G(u)$ denotes the set of neighbours of u in G).

   Give examples of a simple 2-connected graph G and an edge $e \in E(G)$ with the following properties, and explain why your examples are correct:

   (a) $\kappa(G \cdot e) > \kappa(G)$.

   (b) $\kappa(G \cdot e) < \kappa(G)$.

   (c) $\kappa(G \cdot e) = \kappa(G)$, but $\kappa'(G \cdot e) > \kappa'(G)$.

   (d) $\kappa(G \cdot e) = \kappa(G)$, but $\kappa'(G \cdot e) < \kappa'(G)$.

3. Let G be a simple graph with $E(G) \neq \emptyset$, and let D denote the set of vertices of G with maximum degree in G.

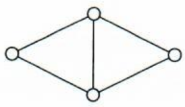
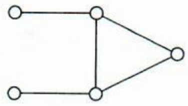
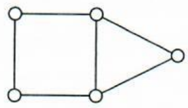
   (a) Prove the following statement: If D is an independent set, then G has a matching that saturates all the vertices of D.

   (b) Show that the implication in (a) does not hold in general if D is not an independent set.

Please turn over.

4. Suppose that the simple graph G has the following properties: G is a connected graph with $\nu \geq 6$ and $\Delta = 4$, and G contains a complete subgraph on 4 vertices. Determine $\chi(G)$.

5. Determine the chromatic polynomial of the following three graphs:

(a)  (b)  (c) 

6. In the cube graph Q_3 (see Figure 1), let u and v denote two vertices at distance 3 of each other. Add only the edge uv. Is the resulting graph planar? Justify your answer.

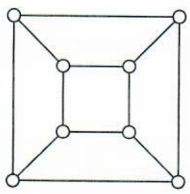


Figure 1: The cube graph Q_3 of exam question 6

Norm:

	1		2				3		4	5			6
Exercise	a	b	a	b	c	d	a	b		a	b	c	
Points	4	2	1.5	1.5	2	2	4	2	5	2	2	3	5

Total: 36 points. Your grade is $\frac{1}{4}\times$ (your total score of points plus 4) (rounded).