Exam Graph Theory (191520751)

University of Twente

21 April 2023, 13:45-16:45

This exam has 7 exercises.
Motivate/justify all your answers by explaining how you got to them.
You may not use any electronic device or lecture materials, books, notes, et cetera.
Don't forget to turn off your cell phone!

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1. Consider the sequence $D = (7, 7, k, 5, 4, 4, 2\ell, 2\ell)$ for some nonnegative integers k and ℓ.

   (a) Show that if there exists a simple graph with degree sequence D, then k is odd and $\ell \geq 1$.

   (b) Find all values of k and ℓ with $5 \leq k \leq 7$ and $1 \leq \ell \leq 2$ for which there exists a simple graph with degree sequence D.

2. Suppose G is a connected graph with a cut edge $e = uv$. Let G_u and G_v denote the components of $G - e$ containing u and v, respectively.

   (a) Determine an expression for $\tau(G)$ in terms of $\tau(G_u)$ and $\tau(G_v)$.

   (b) Compute $\tau(G)$ for the graph G in Figure 1 below.

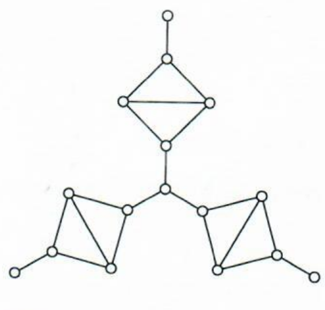

Figure 1: The graph G of exam question 2.

3. Let G be a simple graph and assume that all vertices of G have odd degree. Prove that if $o(G - S) \leq |S| + 1$ for all proper subsets $S \subset V(G)$, then G has a perfect matching.

4. Let G be a simple k-regular graph for some integer $k \geq 2$. Prove that if every 2-factor of G contains at least one odd cycle, then $\chi'(G) = k + 1$.

Please turn over.

5. Let G_1 and G_2 be two simple 4-critical graphs.
Let $e_i = u_i v_i$ be an edge of G_i and let $H_i = G_i - e_i$, for $i = 1, 2$.

   (a) Prove that in every 3-colouring of H_1, u_1 and v_1 receive the same colour.

   (b) Prove that for any edge f_1 of H_1, there exists a 3-colouring of $H_1 - f_1$ in which u_1 and v_1 receive different colours.

   Clearly, the statements in (a) and (b) also hold for the graph H_2.

   Now let G^* be obtained from H_1 and H_2 by identifying v_1 and v_2 and adding an edge u_1u_2; see Figure 2 for an illustration of this construction.

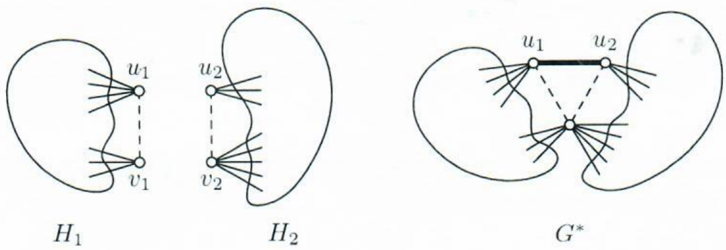

Figure 2: The construction described in exam question 5.

   (c) Prove that G^* is 4-critical.

6. For an integer r, the polynomial $k^5 - rk^4 + (3 - r)k^3$ is the chromatic polynomial of a simple graph G.

   (a) Show that $r > 0$ and $r \leq 3$.

   (b) Show that $r = 2$ and that G is bipartite.

   (c) Determine two nonisomorphic graphs with chromatic polynomial $k^5 - 2k^4 + k^3$.

7. Let G be a simple 2-connected planar graph, and suppose that G has an embedding in which all faces have degree 6. Prove that G is not 3-edge-connected.

Norm:

	1		2		3	4	5			6			7
Exercise	a	b	a	b			a	b	c	a	b	c	
Points	2	3	2	3	4	4	2	3	3	2	2	2	4

Total: 36 points. Your grade is $\frac{1}{4} \cdot$ (your total score of points plus 4), rounded.