Optimal Control
(course code: 156162)

Date: 09-04-2013
Place: CR-2K
Time: 08:45–11:45

1 test

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 - x_1/x_2 \\ 1 - x_1 x_2 \end{bmatrix} \tag{1}$$

   (a) Determine all points of equilibrium
   (b) Determine the linearization at all points of equilibrium
   (c) Determine the type of stability of the nonlinear system at all points of equilibrium

2. Formulate the theorem of LaSalle.

3. Consider functions $x : [0,1] \to \mathbb{R}$ and cost

$$J(x) := \int_0^1 -x^2(t) - \dot{x}^2(t) + 2x(t)\,e^{2t}\,dt$$

   (a) Determine the Euler-Lagrange equation for this problem
   (b) Solve the Euler-Lagrange equation with $x(0) = -1/3, x(1) = 1$
   (c) Are the second order conditions of Legendre satisfied?
   (d) Is the solution $x_*(t)$ found in part (b) of this problem globally maximizing $J(x)$ subject to $x(0) = -1/3, x(1) = 1$?

4. Consider the system

$$\dot{x}_1(t) = \cos(u(t)) \qquad x_1(0) = 0,$$
$$\dot{x}_2(t) = 2\sin(u(t)) \qquad x_2(0) = 0$$

   with cost

$$J(x) = -x_1^2(1) - x_2^2(1).$$

   The input is restricted to $u(t) \in [0,1]$.

   (a) Determine the Hamiltonian
   (b) Determine the Hamiltonian equations for state, costate and input
   (you do not yet have to solve for u)
   (c) Show that the optimal $u_*(t)$ is constant.
   (d) Determine the optimal input $u_*(t)$ and $p_*(t)$.

5. Suppose

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = 1$$

   and that

$$J(x) = \int_0^T x^2(t) + x(t)u(t) + \frac{1}{2}u^2(t)\,dt$$

   for some arbitrary positive T.

   (a) Try a value function of the form $W(x,t) = x^2 P(t)$ and rewrite the resulting Hamilton-Jacobi-Bellman equations as a differential equation in $P(t)$ including a final condition on $P(T)$.
   (b) Does the differential equation for $P(t)$ have a solution on $[0,T]$?
   [Hint: you do not need to solve the differential equation.]

6. Let $Q > 0, R > 0$. Suppose that (A,B) is controllable and let P be the LQ-solution of the algebraic Riccati equation. The optimal input is then static of the form $u_*(t) = Fx_*(t)$, for some matrix F, and so the optimal closed-loop system becomes

$$\dot{x}_*(t) = (A + BF)x_*(t).$$

   Define $V(x) := x^T P x$.

   (a) What is F?
   (b) Show that $\dot{V}(x_*(t)) < 0$ in the closed-loop system for every $x_*(t) \neq 0$
   (c) The origin is an equilibrium of the closed loop. Is $V(x)$ a Lyapunov function for the origin of the closed loop system?
   (Be as precise as possible in your derivation.)

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problem:	1	2	3	4	5	6
points:	2+2+2	3	1+3+2+2	1+2+3+3	3+2	1+3+3

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u), \qquad x(0) = x_0,$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x,p,u), \qquad p(T) = \frac{\partial S}{\partial x}(x(T))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = S$$

Hamilton-Jacobi-Bellman:

$$\frac{\partial W(x,t)}{\partial t} + \min_{v \in \mathbb{U}}\left[\frac{\partial W(x,t)}{\partial x^T} f(x,v) + L(x,v)\right] = 0, \qquad W(x,T) = S(x)$$

1.

(a) $x_1/x_2 = 1$ so $x_1 = x_2$. Also $x_1 x_2 = 1$ so $x_1 = x_2 = \pm 1$: two equilibria $(1,1)$ and $(-1,-1)$

(b) Jacobian is $\left[\begin{smallmatrix} -1/x_2 & -x_1 \\ -x_2 & -x_1 \end{smallmatrix}\right]$. At $\bar{x} = (1,1)$ this gives $\dot{\delta}_x = \left[\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right]\delta_x$ and at $\bar{x} = (-1,-1)$ this gives $\dot{\delta}_x = \left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]\delta_x$

(c) The eigenvalues of $\left[\begin{smallmatrix} -1 & 1 \\ -1 & -1 \end{smallmatrix}\right]$ are $-1 \pm \mathrm{i}$. All real parts are < 0 so asymptotically stable. The eigenvalues of $\left[\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right]$ are $1 \pm \mathrm{i}$. There is an eigenvalue with real part > 0 so unstable.

2.

(a) $0 = \left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right)(-x^2(t) - \dot{x}^2(t) + 2x(t)\mathrm{e}^{2t}) = 2(-x(t) + \ddot{x}(t) + \mathrm{e}^{2t})$.

(b) so $-\ddot{x}(t) + x(t) = \mathrm{e}^{2t}$. Homogeneous solution is $c\,\mathrm{e}^{t} + d\,\mathrm{e}^{-t}$. Particular solution is $-\frac{1}{3}\mathrm{e}^{2t}$. General solution hence is $x(t) = c\,\mathrm{e}^{t} + d\,\mathrm{e}^{-t} - \frac{1}{3}\mathrm{e}^{2t}$. Determine c, d from $-1/3 = x(0) = c + d - 1/3$ and $1 = x(1) = c\,\mathrm{e} + c\,\mathrm{e}^{-1} - 1/3\,\mathrm{e}^{2}$. Hence $c = -d$ and $c(\mathrm{e} + \mathrm{e}^{-1}) = 1 + \mathrm{e}^2/3$...

(c) $\partial^2 F/\partial x^2 = -2$ so not > 0 so not satisfied.

(d) sufficient for global maximality is that the Hessian $H(t, x, \dot{x})$ is negative definite for all $t, x, \dot{x}$. We have $H = \left[\begin{smallmatrix} -2 & 0 \\ 0 & -2 \end{smallmatrix}\right]$. All its eigenvalues are < 0 so H is negative definite and x_* hence globally maximizing.

3.

(a) $H(x, p, u) = p_1 \cos(u) + p_2 2 \sin(u)$.

(b) State equations are as given. Co-state equations are $\dot{p} = -\partial H(x, p, u)/\partial x = \left[\begin{smallmatrix} 0 \\ 0 \end{smallmatrix}\right]$ with final condition $p(1) = \left[\begin{smallmatrix} -2x_1(1) \\ -2x_2(1) \end{smallmatrix}\right]$. The input u_* minimizes pointwise: $H(x_*(t), p_*(t), u_*(t)) \leq H(x_*(t), p_*(t), v)$ for all $v \in [0,1]$

(c) the above shows that costate $p(t)$ is constant.

A bit sloppy: "it is known that the Hamiltonian

$$p_1 \cos(u_*(t)) + p_2 2 \sin(u_*(t))$$

is constant over time (correct) and since p_1, p_2 are constant the $u(t)$ will be constant as well."

Less sloppy: at optimality p_1 and p_2 are not both zero (otherwise $J = 0$) so the Hamiltonian is a nonzero sinusoid $A\cos(u_*(t) + \phi)$. Pontryagin says that $u_*(t)$ minimizes this sinusoid. The minimum over $[0,1]$ of this sinusoid is either attained at one or both of the boundaries $0, 1$, or at a unique minimizer (hence constant over time) in $(0,1)$. So if the Hamiltonian at $u_* = 0$ differs from that at $u_* = 1$ the minimizing input $u_*(t)$ is constant[1]

(d) since $u_*(t) = u_*$ is constant we have $x_1(1) = \cos(u_*), x_2(1) = 2\sin(u_*)$. So $J = -\cos(u_*)^2 - 4\sin^2(u_*) = 1 - 3\sin^2(u_*)$ which is minimal over $u_* \in [0,1]$ for $u_* = 1$. Then $J = -1 - 3\sin^2(1)$ and $p_1(t) = -2\cos(1), p_2(t) = -2\sin(1)$

4.

(a) For $W(x, t) = P(t)x^2$ HJB becomes

$$x^2 \dot{p} + \min_{v \in \mathbb{R}} (2xp(x + v) + x^2 + xv + \frac{1}{2}v^2) = 0$$

(with $p(T) = 0$). Since function to be minimized is "positive definite parabola" minimizer is the solution of $0 = \frac{\partial}{\partial v}(2xp(x + v) + x^2 + xv + \frac{1}{2}v^2) = 2xp + x + v$ so $v = -x(1 + 2p)$. Insert this into HJB:

$$x^2 \dot{p} + (2xp(-2xp) - x^2(1 + 2p) + \frac{1}{2}x^2(1 + 2p)^2 = 0$$

Division by x^2 and work out the products:

$$\dot{p} - 2p^2 + 1/2 = 0, \qquad p(T) = 0.$$

(b) The RDE is $\dot{p}(t) = 2p^2(t) - 1/2$. Which is a standard RDE for $A = 0, B = 1, R = 1/2, Q = 1/2, S = 0$. Lecture notes says: if $Q, S \geq 0, R > 0$ then RDE has solution on $[0, T]$. That is the case so $p(t)$ exists on $[0, T]$

5.

(a) $F = -R^{-1}B^{\mathrm{T}}P$

(b) $V(x)$ is the cost-to-go for our optimal $u = Fx$. So according to chapter 1 we have $\dot{V}(x) = -L(x) = -(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)$. Since $u^{\mathrm{T}}Ru \geq 0$ we have $\dot{V}(x) \leq -x^T Qx < 0$ for every $x \neq 0$ (since $Q > 0$).

Alternative derivation:

$$\begin{aligned} \dot{V}(x) &= \dot{x}^{\mathrm{T}}Px + xP\dot{x} \\ &= x^{\mathrm{T}}(A + BF)^{\mathrm{T}}Px + x^{\mathrm{T}}P(A + BF)x \\ &= x^{\mathrm{T}}(A^{\mathrm{T}}P + PA - 2PB^{\mathrm{T}}R^{-1}BP)x \\ &= x^{\mathrm{T}}(-Q - PB^{\mathrm{T}}R^{-1}BP)x \\ &= -x^{\mathrm{T}}Qx - u^{\mathrm{T}}Ru. \end{aligned}$$

(c) Clearly $V(x) := x^T Px$ is C^1 and $\dot{V}(x) < 0$ for all $x \neq 0$ in the closed loop system. If $P > 0$ then $V(x)$ is a positive definite function relative to 0 so then it is a Lyapunov function and stability of the closed loop follows.

So why is $P > 0$? If P is singular then $V(x_0) = x_0^{\mathrm{T}}Px_0 = 0$ for some vector $x_0 \neq 0$. But as $\dot{V}(x_0) < 0$ for this x_0 we would have that $x(t)^{\mathrm{T}}Px(t)$ is < 0 for $t > 0$. Not possible because $P \geq 0$ (says Riccati theory). So P nonsingular and $P \geq 0$. Therefore $P > 0$.

[1] If the minimum is attained at both $u_* = 0$ and $u_* = 1$ then the optimal $u_*(t)$ might switch between 0 and 1 throughout $t \in [0,1]$. Such switching cannot be optimal though...