Optimal Control
(course code: 191561620)

Date: 07-04-2015
Place: Sportcentrum
Time: 08:45-11:45

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -x_1 + x_2 \\ x_1 - x_2 - x_1 x_3 \\ -x_3 + x_1 x_2 \end{bmatrix}. \tag{1}$$

   (a) Determine all points of equilibrium.

   (b) Consider equilibrium $\bar{x} = (0,0,0)$. What does candidate Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$ allow us to conclude about the stability properties of this equilibrium?

   (c) What does LaSalle allow us to conclude about the stability properties of this equilibrium $\bar{x} = (0,0,0)$ with Lyapunov function $V(x) = x_1^2 + x_2^2 + x_3^2$?

2. In Chapter 2 there is a convexity result that can be used to check for global optimality of solutions of the simplest problem in the calculus of variations. Formulate this result as explicitly as you can. (You do not NOT have to prove the result.)

3. Consider the cost function and boundary conditions

$$\int_0^1 (x(t) - t)^2 + (\dot{x}(t) - t)^2 \, dt, \qquad x(0) = 1, x(1) = 2.$$

   Determine the function $x_*(t)$ that minimizes this cost function with $x_*(0) = 1, x_*(1) = 2$.

4. Let $a \in \mathbb{R}$ and $q > 0$. Consider the infinite horizon LQ problem of minimizing

$$\int_0^\infty q x^2(t) + u^2(t) \, dt$$

   with $\dot{x}(t) = a x(t) + u(t), x(0) = 1$.

   (a) Determine all solutions of the appropriate algebraic Riccati equation.

   (b) Determine the optimal input $u = Fx$ (that is find the number F) and determine the optimal cost.

   (c) Discuss what happens with the optimal control, optimal state and optimal cost if q is huge and explain in words why this makes sense.

5. Consider the system

$$\dot{x}(t) = u(t), \qquad x(0) = x_0$$

with $u(t)$ free to choose, $u(t) \in \mathbb{R}$, and with cost

$$J_{[0,1]}(x_0, u(\cdot)) = x(1) + \int_0^1 x(t) + \tfrac{1}{2}u^2(t)\, dt.$$

(a) Determine the Hamiltonian

(b) Determine the costate $p(t)$ as a function of time

(c) Verify that the Hamiltonian is constant over time for the optimal state, costate and input

6. Consider exactly the same optimal control problem:

$$\dot{x}(t) = u(t), \qquad x(0) = x_0$$

with $u(t)$ free to choose, $u(t) \in \mathbb{R}$, and with cost

$$J_{[0,1]}(x_0, u(\cdot)) = x(1) + \int_0^1 x(t) + \tfrac{1}{2}u^2(t)\, dt.$$

(a) Determine the Hamilton-Jacobi-Bellman (HJB) equation for $V(x,t)$ and express the minimizing u in the HJB equation in terms of $V(x,t)$.

(b) Try a $V(x,t)$ of the form $V(x,t) = P(t)x + Q(t)$ and then determine $P(t)$ and $Q(t)$ explicitly.

(c) Is the solution $V(x,t)$ of the HJB equations the value function of the optimal control problem for every $x \in \mathbb{R}$ and $t \in [0,1]$?

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problem:	1	2	3	4	5	6
points:	2+3+3	3	4	1+3+2	1+2+4	2+4+2

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange:

$$\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \left(\frac{\partial F}{\partial \dot{x}}\right)\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H(x,p,u),}{\partial p} \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H(x,p,u),}{\partial x}, \qquad p(T) = \frac{\partial S(x(T))}{\partial x}$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = S$$

Hamilton-Jacobi-Bellman:

$$\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}} \left[\frac{\partial V(x,t)}{\partial x^T} f(x,u) + L(x,u)\right] = 0, \qquad V(x,T) = S(x)$$