Optimal Control
(course code: 191561620)

Date: 12-04-2017
Place: Sports centre "Hal 1"
Time: 08:45-11:45

1. Consider

$$\dot{x}_1 = -3x_1 + 2x_2$$
$$\dot{x}_2 = -x_1$$

with equilibrium $\bar{x} = (0,0)$. Determine a Lyapunov function $V(x)$ such that $\dot{V}(x) = -x_1^2 - x_2^2$ and verify that this $V(x)$ is a strong Lyapunov function for this system.

2. Formulate LaSalle's invariance principle.

3. Consider the cost function

$$\frac{1}{2}x(1) + \int_0^1 (\dot{x}(t))^4 \, \mathrm{d}t.$$

(a) Minimize this cost over all $x(t)$ subject to $x(0) = 1$ and $x(1) = 0$.

(b) Is Legendre's second-order condition for optimality satisfied?

(c) Minimize this cost over all $x(t)$ subject to $x(0) = 1$ but with a free endpoint $x(1)$.

4. Consider

$$\dot{x}_1(t) = -x_1(t) + u(t), \qquad x_1(0) = 0, \quad x_1(1) = 1/2,$$
$$\dot{x}_2(t) = x_1(t), \qquad x_2(0) = 0.$$

Here $u(t)$ is the flow of water in the first reservoir, $x_1(t)$ is the level in the first reservoir and $x_2(t)$ is the level in the second reservoir. The inflow $u(t)$ can not be negative and can be at most one:

$$u(t) \in [0,1].$$

We want to *maximize* $x_2(1)$ subject to the initial conditions $x_1(0) = x_2(0) = 0$ and a final condition on the first reservoir $x_1(1) = 1/2$.

(a) Determine the cost function J and the Hamiltonian H.

(b) Determine the costate equations and its general solution $p(t)$.

(c) How often on $t \in [0,1]$ does the optimal $u_*(t)$ switch from 1 to 0? How often on $t \in [0,1]$ does the optimal $u_*(t)$ switch from 0 to 1?

(d) Sketch the graph for $t \in [0,1]$ of the optimal input $u_*(t)$ and the optimal $x_1(t)$ and $p_1(t)$. (A sketch suffices because an exact formula for the switching time(s) may be hard to find.)

5. Suppose that

$$\dot{x}(t) = u(t)x(t), \qquad x(0) = x_0 > 0$$

and that

$$u(t) \in [0,1]$$

for all time and that the cost function is

$$J_{[0,3]}(x_0, u(\cdot)) = x(3) + \int_0^3 (u(t)-1)x(t)\,\mathrm{d}t.$$

(a) Try as value function a function of the form $V(x,t) = q(t)x$ and with it determine the Hamilton-Jacobi-Bellmann equations.

(b) Express the candidate optimal $u_*(t)$ as a function of $q(t)$ (Hint: $x(t)$ is always positive.)

(c) Determine $q(t)$ for all $t \in [0,3]$.

(d) Determine the optimal $u_*(t)$ explicitly as a function of time and argue that this is the true optimal control (so not just the "candidate" optimal control).

(e) What is the optimal cost $J_{[0,3]}(x_0, u_*(\cdot))$?

6. Consider the optimal control problem

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = x_0$$

with $u(t) \in \mathbb{R}$ and cost

$$J_{[0,\infty)}(x_0, u(\cdot)) = \int_0^\infty 3x^2(t) + u^2(t)\ \mathrm{d}t.$$

(a) Determine the corresponding Algebraic Riccati Equation.

(b) Determine the optimal input $u(t)$ as a function of $x(t)$.

(c) Determine the optimal cost.

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problem:	1	2	3	4	5	6
points:	4	3	2+2+2	2+2+2+2	2+2+3+2+2	1+2+1

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange eqn: $\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$

Beltrami identity: $F - (\frac{\partial F}{\partial \dot{x}})\dot{x} = C$

Standard Hamiltonian eqn: $\dot{x} = \frac{\partial H(x,p,u),}{\partial p} \qquad x(0) = x_0,$

$\dot{p} = -\frac{\partial H(x,p,u),}{\partial x} \qquad p(T) = \frac{\partial S(x(T))}{\partial x}$

LQ Riccati differential eqn: $\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \quad P(T) = S$

HJB eqn: $\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}} \left[\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u)\right] = 0, \quad V(x,T) = S(x)$

1. So we need to find a matrix P such that $PA + A^{\mathrm{T}}P = -I$ for then $\dot{V}(x) = -x^{\mathrm{T}}Ix = -x_1^2 - x_2^2$.

$$\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix}\begin{bmatrix}-3 & 2\\ -1 & 0\end{bmatrix} + \begin{bmatrix}-3 & -1\\ 2 & 0\end{bmatrix}\begin{bmatrix}\alpha & \beta\\ \beta & \gamma\end{bmatrix} = \begin{bmatrix}-1 & 0\\ 0 & -1\end{bmatrix}$$

This gives

$$P = \frac{1}{4}\begin{bmatrix}1 & -1\\ -1 & 5\end{bmatrix}$$

Since $P_{11} > 0$ and $\det(P) = 1/4 > 0$ it is positive definite hence a strong Lyapunov function.

2. See lecture notes.

3. (a) Notice that the terminal cost $\frac{1}{2}x(1)$ plays no role here because we fix $x(1) = 0$. Euler-Lagrange gives $0 = -\frac{\mathrm{d}}{\mathrm{d}t}4\dot{x}^3 = -12\dot{x}^2\ddot{x}$. So either x is constant or linear at any moment in time. Hence $x(t) = at + b$. Given the initial $x(0) = 1$ and $x(1) = 0$ this gives $x(t) = 1 - t$.

   (b) $(\partial^2 F)/(\partial\dot{x}^2) = 12\dot{x}^2$ so ≥ 0. So, yes, it is satisfied.

   (c) Now the free end-point enters our story: $\partial F/\partial\dot{x} + \partial S/\partial x$ should be zero at the final time. This gives $4\dot{x}^3 + \frac{1}{2} = 0$ so $\dot{x} = -1/2$ at the final time. Since EL says $x(t) = at + b$ it means that $\dot{x} = -1/2$ all the time: $x(t) = 1 - t/2$.

   If you forgot the free endpoint formula you can also use that $x(t) = 1 - ct$ (because of Euler-Lagrange and $x(0) = 1$) and then minimize the resulting cost $J = (1-c)/2 + c^4$ over all c which, again, is minimal iff $c = 1/2$, so $x(t) = 1 - t/2$.

4. (a) There are two choices. Either you pick $J_a = -x_2(1)$ and then $H = p_1(-x_1 + u) + p_2 x_1$ or you pick $J_b = \int x_1$ and then $H_b = p_1(-x_1 + u) + p_2 x_1 + x_1$.

   (b) The choice also affects the co-state equations. For J_a, H_a we get

$$\dot{p}_1 = p_1 - p_2, \quad \dot{p}_2 = 0, \quad p_2(1) = -1$$

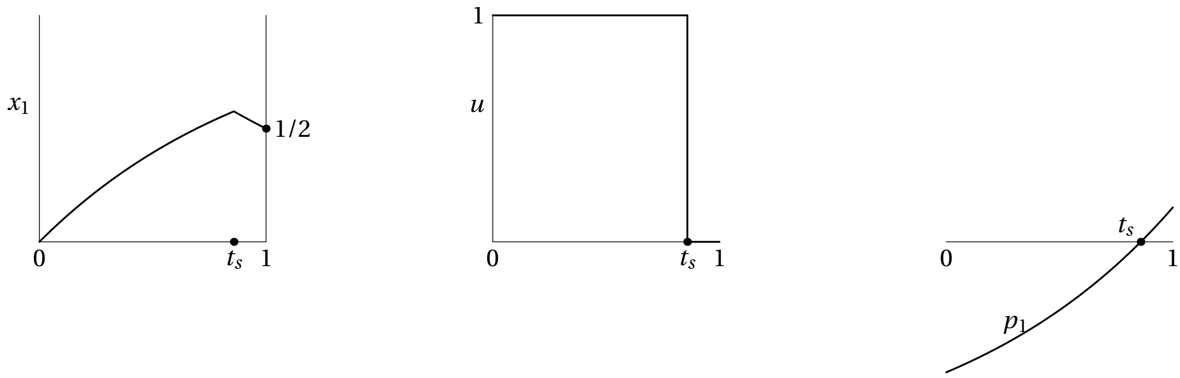
   The general solution is $p_2(t) = -1$ for all time and $p_1(t) = -1 + c\mathrm{e}^t$
   For the other you get $\dot{p}_1 = p_1 - p_2 - 1, \dot{p}_2 = 0, p_2(1) = 0$ and then the general solution is $p_2(t) = 0$ for all t and the same p_1 as in the other case: $p_1(t) = -1 + c\mathrm{e}^t$.

   (c) The optimal u minimizes the Hamiltonian so

$$u(t) = \begin{cases}1 & \text{if } p_1(t) < 0\\ 0 & \text{if } p_1(t) > 0\end{cases}$$

   Since $p_1(t) = -1 + c\mathrm{e}^t$ the p_1 can switch sign at most once (from negative to positive). So u can switch at most once (from 1 to 0).

(d) So $u(t) = 1$ on some $[0, t_s]$ and then 0 on $[t_s, 1]$. On $[0, t_s]$ we then have $\dot{x} = -x + 1$ so $x_1 = 1 - \mathrm{e}^{-t}$ which grows in the direction of 1 and then on $[t_s, 1]$ the $x_1(t)$ satisfies $\dot{x}_1 = -x_1$ so decays exponentially. This gives something like:



Explanation: there is a unique t_s for which $x_1(1) = 1/2$. Then c in $p_1(t) = -1 + c\,\mathrm{e}^t$ is such that $p_1(t)$ switches sign at this t_s.

(Actually, the switching time t_s *can* be calculated. It is $\ln(\mathrm{e}/2 + 1)$ which is 0.8583.. and then $c = 1/(\mathrm{e}/2 + 1)$.)

5. (a) $x\dot{q}(t) + \min_{u\in[0,1]}(q(t)ux + (u-1)x) = 0, q(3)x = x$

   (b) Since $x > 0$ we may cancel x from HBJ to obtain $\dot{q}(t) + \min_{u\in[0,1]}(q(t)u + u - 1) = 0$. The $u(t) = 0$ if $q(t) + 1 > 0$ and $u(t) = 1$ if $q(t) + 1 < 0$

   (c) since $q(3) = 1$ we have $q(t) + 1 > 0$ near the final time. So then $u = 0$ which turns HJB into $\dot{q}(t) - 1 = 0, q(3) = 1$. So then $q(t) = t - 2$. This is the solution on $[1, 3]$ for then we still have $q(t) + 1 > 0$. On $[0, 1]$ we then get $u(t) = 1$ so then HJB becomes $\dot{q}(t) + q(t) = 0$ which given $q(1) = -1$ gives $q(t) = -\mathrm{e}^{1-t}$.

   (d) $u(t) = 1$ on $[0, 1]$ and zero on $[1, 3]$. Then $x(t)$ satisfies $\dot{x} = xu$ which is well defined for all $t \in [0, 3]$. Then Chapter 4 says that the "candidate" is truly optimal

   (e) and the optimal cost is $V(x_0, 0) = q(0)x_0 = -\mathrm{e}\,x_0$.

6. (a) $-PA - A'P + PBR^{-1}B'P - Q = P^2 - 2p - 3 = 0$

   (b) $P^2 - 2P - 3 = (P-3)(P+1)$ so $P = 3$ and $u = -R^{-1}B'P = -3x$

   (c) $V(x_0, 0) = 3x_0^2$.