Optimal Control
(course code: 191561620)

Date:	23-01-2019
Place:	Therm
Time:	08:45–11:45 (till 12:30 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	NONE

1. Consider the nonlinear differential equation

$$\dot{x}_1(t) = x_2(t),$$
$$\dot{x}_2(t) = -x_1(t) - x_1^2(t)x_2(t).$$

   (a) Determine all equilibria.

   (b) Linearize the system around equilibrium $\bar{x} = (0,0)$.

   (c) What can be concluded on the basis of this linearization about the stability properties of the nonlinear system?

   (d) Is the nonlinear system asymptotically stable around $\bar{x} = (0,0)$?

2. Consider minimization of

$$\int_0^1 \dot{x}^2(t) - x(t)\,\mathrm{d}t, \qquad x(0) = 0, x(1) = 1,$$

over all functions $x : [0,1] \to \mathbb{R}$.

   (a) Solve the Euler-Lagrange equation, and such that $x(0) = 0, x(1) = 1$.

   (b) Show that the obtained solution is optimal.

3. Consider the system

$$\dot{x}(t) = -\alpha x(t) + u(t), \quad x(0) = x_0 \in \mathbb{R}, \qquad \mathbb{U} = [-M, M],$$

for some $\alpha \in \mathbb{R}$ and $M > 0$, and with cost

$$J(u(\cdot)) = \int_0^1 \frac{1}{2} u^2(t)\,\mathrm{d}t + x(1).$$

   (a) Determine the Hamiltonian for this optimal control problem, and the differential equation for the co-state $p(t)$.

   (b) Determine the optimal input. (Distinguish between the case $\alpha \geq 0$ and $\alpha < 0$.)

4. Consider the system $\dot{x}(t) = u(t), x(0) = x_0 \in \mathbb{R}$ with $u(t) \in \mathbb{R}$, and cost

$$\int_0^1 x^2(t) + u^2(t) - 2\beta x(t)u(t)\,\mathrm{d}t + x^2(1).$$

   (a) Derive the Hamilton-Jacobi-Bellman equation.

   (b) Substitute in the Hamilton-Jacobi-Bellman equation the candidate value function $V(x,t)$ of the form $V(x,t) = x^2 P(t)$ and derive an ordinary differential equation in the unknown function $P(t)$ with boundary condition.

   (c) What is the optimal control $u_*(t)$ in terms of $P(t)$ and $x(t)$?

5. Consider the infinite-horizon LQ optimal control problem

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t), \qquad x(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

with $u(t) \in \mathbb{R}$ and cost

$$J_{[0,\infty)}(x_0, u(\cdot)) = \int_0^\infty x_1^2(t) + x_2^2(t) + u^2(t)\,\mathrm{d}t.$$

   (a) Determine the Algebraic Riccati Equation in the unknown $P \in \mathbb{R}^{2\times 2}$, and denote this matrix as

$$P = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}.$$

   (b) Determine a positive semi-definite solution P of the Algebraic Riccati Equation .

   (c) Is the P found in the previous part also the solution P of the infinite horizon LQ problem?

6. The standard Minimum Principle is for *fixed* final time T. What extra condition is added to the Minimum Principle if we also optimize over the final time?

---

problem:	1	2	3	4	5	6
points:	1+2+2+3	4+2	3+5	2+3+1	2+3+1	2

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange eqn: $\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$

Beltrami identity: $F - (\frac{\partial F}{\partial \dot{x}})\dot{x} = C$

Standard Hamiltonian eqn: $\dot{x} = \frac{\partial H(x,p,u)}{\partial p},\ x(0) = x_0 \quad \& \quad \dot{p} = -\frac{\partial H(x,p,u)}{\partial x},\ p(T) = \frac{\partial S(x(T))}{\partial x}$

LQ Riccati differential eqn: $\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \quad P(T) = S$

HJB eqn: $\frac{\partial V(x,t)}{\partial t} + \min_{u\in\mathbb{U}}\left[\frac{\partial V(x,t)}{\partial x^{\mathrm{T}}} f(x,u) + L(x,u)\right] = 0, \quad V(x,T) = S(x)$