Optimal Control
(course code: 191561620)

Date:	20-01-2021
Place:	At home!
Time:	09:00–12:00 (till 12:45 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	see below

Integrity statement. Please read the following paragraph carefully.

By testing you remotely in this fashion, we express our trust that you will adhere to the ethical standard of behaviour expected of you. This means that we trust you to answer the questions and perform the assignments in this test to the best of your own ability, without seeking or accepting the help of any source that is not explicitly allowed by the conditions of this test.

The only allowed sources for this test are:

- the lecture notes "Optimal Control" (pdf or printed)
- the slides (pdf or printed)
- electronic devices, but only to be used:
  - for downloading the test and afterwards uploading your work to Canvas
  - to show the test/book/slides on your screen
  - to write the test (in case you prefer to use a tablet instead of paper to write on)

P.T.O.

A. **Copy the following text verbatim to the first page of your work (handwritten) and sign it. If you fail to do so, your test will not be graded:**

> *I will make this test to the best of my own ability, without seeking or accepting the help of any source not explicitly allowed by the conditions of the test.*

B. **Which Msc programme (AM, BME, EE, SC, ?) do you follow?**

C. **Are you entitled to extra time? (We will check this with CES.)**

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1. Consider the nonlinear system

$$\begin{aligned}\dot{x}_1(t) &= -x_1^3(t) - x_2(t)\\ \dot{x}_2(t) &= x_1^2(t) - x_2^2(t)\end{aligned}$$

   (a) Determine all points of equilibriums $\bar{x} \in \mathbb{R}^2$.

   (b) What can be concluded about the stability property of the nonlinear system around equilibrium $\bar{x} = \left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ based on the linearization?

   (c) There is an equilibrium $\bar{x}$ for which the linearization, $\delta_x(t) = A\delta_x(t)$, has the property that $A + A^{\mathrm{T}}$ is a negative definite matrix. Which equilibrium is that? And what does this mean for the stability properties of the nonlinear system at that equilibrium?

2. Consider the following cost and boundary conditions

$$\int_{-1}^{1} t^2\dot{x}^2(t) + 12x^2(t)\,\mathrm{d}t, \qquad x(-1) = -1, \qquad x(1) = 1.$$

   (a) Determine numbers a, k such that $x_*(t) = at^k$ satisfies the Euler-Lagrange equation, and satisfies the given boundary conditions.

   (b) Does the solution found in (a) satisfy the Legendre necessary condition for optimality?

   (c) Is the solution found in (a) a global optimal solution? If yes, explain why. If it is not optimal or if it is inconclusive, then explain why.

3. Consider the system with negative controls

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = \mathrm{x}_0, \qquad \mathbb{U} = (-\infty, 0]$$

$$J(\mathrm{x}_0, u) = -\frac{x(T)}{\mathrm{e}^2} + \int_0^5 \mathrm{e}^{u(t)} - 1\,\mathrm{d}t.$$

   (a) Determine the Hamiltonian.

   (b) Determine the costate $p_*(t)$ explicitly as a function of time.

   (c) Determine an optimal input (assuming one exists) explicitly as a function of time.

4. Consider the optimal control problem

$$\dot{x}(t) = x(t)(1-u(t)), \quad x(0) = \mathrm{x}_0 > 0, \quad \mathbb{U} = [0,\infty)$$

and cost

$$J(\mathrm{x}_0, u) = -\sqrt{x(T)} + \int_0^T -\sqrt{x(t)u(t)}\,\mathrm{d}t.$$

Throughout we assume that $\mathrm{x}_0 > 0$.

(a) It is given that $\mathrm{x}_0 > 0$. Argue that $x(t) \geq 0$ for all time if u is bounded.

(b) Assume that the HJB equation has a solution of the form

$$V(\mathrm{x}, t) = -\sqrt{Q(t)\mathrm{x}}$$

for some function $Q(t)$. Derive an ordinary differential equation for $Q(t)$ including final condition. The differential equation must not depend on x.

(c) It can be shown that

$$Q(t) = 2\,\mathrm{e}^{T-t} - 1.$$

Knowing that, determine an optimal control $u_*(t)$ in terms of $Q(t), x(t)$, and show that u_* is optimal (so not just a *candidate* optimal control).

(d) Determine the optimal cost $J(\mathrm{x}_0, u_*)$.

5. Consider the linear system

$$\dot{x}(t) = ax(t) + u(t), \qquad x(0) = \mathrm{x}_0 \in \mathbb{R}, \qquad \mathbb{U} = \mathbb{R}$$

with infinite horizon cost

$$J(\mathrm{x}_0, u) = \int_0^\infty \frac{1}{2}x^2(t) + \frac{1}{2}u^2(t)\,\mathrm{d}t.$$

Here a is an arbitrary real number.

(a) Determine F such that $u(t) = -Fx(t)$ solves the infinite-horizon LQ problem.

(b) Suppose $\mathrm{x} \neq 0$. Show that the optimal cost is less than $\mathrm{x}_0^2/2$ if-and-only-if $a < 0$, and explain in words why it is not a surprise that the optimal cost is "small enough" if-and-only-if a is "negative enough".

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problem:	1	2	3	4	5
points:	2+2+3	2+2+2	1+3+5	1+5+2+1	2+3

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange eqn: $\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$

Beltrami identity: $F - \dot{x}^{\mathrm{T}}(\frac{\partial F}{\partial \dot{x}}) = C$

Standard Hamiltonian eqn: $\dot{x} = \frac{\partial H(x,p,u)}{\partial p},\ x(0) = x_0 \quad \& \quad \dot{p} = -\frac{\partial H(x,p,u)}{\partial x},\ p(T) = \frac{\partial K(x(T))}{\partial x}$

HJB eqn: $\frac{\partial V(\mathrm{x},t)}{\partial t} + \min_{\mathrm{u} \in \mathbb{U}}\left[\frac{\partial V(\mathrm{x},t)}{\partial \mathrm{x}^{\mathrm{T}}} f(\mathrm{x}, u) + L(\mathrm{x}, u)\right] = 0, \quad V(\mathrm{x}, T) = K(\mathrm{x})$

LQ Riccati differential eqn: $\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \quad P(T) = S$