Optimal Control (course code: 191561620)

Date:	25-01-2023
Place:	SC2
Time:	08:45–11:45 (till 12:30 for students with special rights)
Course coordinator:	G. Meinsma
Allowed aids during test:	NONE

0. Which Msc programme (AM, BME, EE, SC, ?) do you follow?

1. Consider the nonlinear real system

$$\dot{x}_1(t) = -2x_1(t) + x_1^2(t) - x_2^2(t)$$
$$\dot{x}_2(t) = -2x_2(t) + 2x_1(t)x_2(t).$$

   (a) Determine all points of equilibrium.

   (b) Determine the linearization at all equilibria.

   (c) Determine the type of stability of the system at all equilibria. (So indicate for each equilibrium if the system at that equilibrium is stable, asymptotically stable, globally asymptotically stable or unstable.)

2. Formulate Lyapunov's stability theorem from Appendix B of the lecture notes. (The one about stability using Lyapunov function.) Distinguish a "Lyapunov function" and a "strong Lyapunov function".

3. Consider the calculus of variations problem with free end-point

$$J(x) := \int_0^1 \dot{x}^4(t)\,\mathrm{d}t + \frac{1}{2}x(1), \qquad x(0) = 1.$$

   (a) Determine all solutions of the Euler-Lagrange equation of this problem.

   (b) It can be shown that a smooth optimal solution x_* of the above free end-point calculus of variations problem exists. Determine this solution x_*.

   (c) Does Legendre's necessary condition hold for the x_* found in part (b) of this problem?

4. Let $T > 0$. Consider the system and class of inputs

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = \mathrm{x}_0, \qquad \mathbb{U} = \mathbb{R}$$

   and with cost

$$J(u) = \frac{1}{2}x^2(T) + \int_0^T -x^2(t) - x(t)u(t)\,\mathrm{d}t.$$

   (a) Determine the Hamiltonian.

   (b) Determine the costate differential equation.

   (c) Show that every continuous input $u(t)$ satisfies the Hamiltonian equations (for state and co-state).

   (d) Show that for every continuous input we have $J(u) = \frac{1}{2}\mathrm{x}_0^2$.

5. Consider the optimal control problem

$$\dot{x}(t) = x(t)(1-u(t)), \quad x(0) = \mathrm{x}_0 > 0, \quad \mathbb{U} = [0,\infty)$$

and cost

$$J(\mathrm{x}_0, u) = -\sqrt{x(T)} + \int_0^T -\sqrt{x(t)u(t)}\,\mathrm{d}t.$$

Throughout we assume that $\mathrm{x}_0 > 0$.

(a) It is given that $\mathrm{x}_0 > 0$. Argue that $x(t) \geq 0$ for all time if u is bounded.

(b) Assume that the HJB equation has a solution of the form

$$V(\mathrm{x}, t) = -\sqrt{Q(t)\mathrm{x}}$$

for some function $Q(t)$. Derive an ordinary differential equation for $Q(t)$ including final condition. The differential equation must not depend on x and u.

(c) It can be shown that

$$Q(t) = 2\,\mathrm{e}^{T-t} - 1.$$

Knowing that, determine an optimal control $u_*(t)$ in terms of $Q(t), x(t)$, and show that u_* is optimal (so not just a *candidate* optimal control).

(d) Determine the optimal cost $J(\mathrm{x}_0, u_*)$.

6. Let $R > 0$. Consider the infinite horizon LQ problem-with-stability with

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = 1, \qquad J_{[0,\infty)}(\mathrm{x}_0, u) = \int_0^\infty 2x^2(t) + \tfrac{1}{2}Ru^2(t)\,\mathrm{d}t.$$

(a) Determine the optimal input $u_*(t)$ as a state-feedback as well as an explicit function of time.

(b) As R goes to infinity the quadratic term $Ru^2(t)$ in the cost grows and grows. So one would expect that the optimal control $u_*^2(t)$ becomes very small as $R \to \infty$, and/or that the optimal cost goes up as $R \to \infty$. What happens as $R \to \infty$ (discuss both $u_*(t)$ and cost) and also explain in words why this behavior is to be expected.

problem:	1	2	3	4	5	6
points:	2+2+2	2	2+2+1	1+2+3+2	1+5+2+1	3+3

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange eqn: $\left(\frac{\partial}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$

Beltrami identity: $F - \dot{x}^{\mathrm{T}}\left(\frac{\partial F}{\partial \dot{x}}\right) = C$

Standard Hamiltonian eqn: $\dot{x} = \frac{\partial H(x,p,u)}{\partial p}, \; x(0) = \mathrm{x}_0 \quad \& \quad \dot{p} = -\frac{\partial H(x,p,u)}{\partial x}, \; p(T) = \frac{\partial K(x(T))}{\partial x}$

HJB eqn: $\frac{\partial V(\mathrm{x},t)}{\partial t} + \min_{\mathrm{u} \in \mathbb{U}} \left[\frac{\partial V(\mathrm{x},t)}{\partial \mathrm{x}^{\mathrm{T}}} f(\mathrm{x}, u) + L(\mathrm{x}, u)\right] = 0, \quad V(\mathrm{x}, T) = K(\mathrm{x})$

LQ Riccati differential eqn: $\dot{P}(t) = -P(t)A - A^{\mathrm{T}}P(t) + P(t)BR^{-1}B^{\mathrm{T}}P(t) - Q, \quad P(T) = S$