Exam Optimal Control code 156162

Date : 10-04-2008
Location : SP 4
Time : 13.30-16.30

Please provide clear motivation for all your answers and indicate which theorems you are using.
Do not spend too much time on a single item. If you are not able to solve part(s) of a problem, then move on and use those parts as if you have already solved them.

There are three exercises.

1. Consider the system of differential equations

$$\begin{aligned} \tfrac{d}{dt}x_1 &= x_2, \\ \tfrac{d}{dt}x_2 &= -x_1 - (\alpha + x_1^2)x_2. \end{aligned}$$

   (a) Determine the equilibrium points.

   (b) Use the linearization theorem to classify the stability properties of the equilibrium $x^* = 0$ for all $\alpha \neq 0$. ~~For $\alpha = 0$.~~

   (c) What can you say on the basis of the linearization for the case $\alpha = 0$?

   (d) Find a suitable Lyapunov function to show that the equilibrium is stable when $\alpha = 0$.

   (e) Is the origin *asymptotically* stable for $\alpha = 0$?

2. Consider the following minimization problem:

$$J = \int_0^4 x(t) + \dot{x}(t)^2 dt, \quad x(0) = 0, \quad x(4) = 1$$

   (a) Find the (unique) candidate solution of the minimization problem and calculate the corresponding minimal value of J.

   In what follows we have to satisfy the additional requirement that $\dot{x}(t) \geq 0$ and $x(t) \geq 0$.

   (b) Does your solution of 2a satisfy the constraints?

   (c) A pragmatic attitude would be to cut off the solution of 2a at the initial interval, that is, take $x(t) = 0$ for those ts for which the solution $x(t)$ in 2a is negative. Calculate the corresponding value of J.

Z.0.2.

(d) Define $u = \dot{x}$ and write the problem as an optimal control problem with a constraint on u and x.

(e) Give the Hamiltonian for the optimal control problem and write the candidate optimal control as a function of the co-state p. Make sure that u satisfies the additional constraint.

(f) What can be said about the constraint on x if the constraint on u is satisfied?

(g) Determine the general solution for p. Argue that $p(t)$ changes sign at most once. Call the time instant at which $p(t)$ changes sign t_1.

(h) Determine the optimal control, the corresponding state trajectory and the optimal value of J.

(i) Comment on the three different values of J corresponding to the unconstrained problem, the pragmatic solution to the constrained problem, and the optimal solution to the constrained problem respectively. Explain how they are ordered.

3. Consider the system and cost criterion:

$$\frac{d}{dt}x = u \quad x(0) = 1 \quad J(x_0, u) = x(1) + \int_0^1 x(t) + \frac{1}{2}u(t)^2 dt.$$

(a) Provide the Hamiltonian and the differential equations for state and co-state.

(b) Determine the candidate optimal control u and the corresponding state trajectory x.

Subsequently, consider the same system and cost criterion, but now with starting time t_0, $0 \leq t_0 \leq 1$. The initial state is denoted by $x(t_0) = x_0$.

$$\frac{d}{dt}x = u \quad x(t_0) = x_0 \quad J(x_0, u, t_0) = x(1) + \int_{t_0}^1 x(t) + \frac{1}{2}u(t)^2 dt.$$

(c) Again, determine the candidate optimal control u and the corresponding state trajectory x.

(d) Give the definition of the notion of *value function* V.

(e) Show that:

$$V(x_0, t_0) = (2 - t_0)x_0 + \frac{1}{6}t_0^3 - t_0^2 + 2t_0 - \frac{7}{6}$$

(f) Provide the corresponding Hamilton-Jacobi-Bellman equation and determine its solution.