Optimal Control
(course code: 156162)

Date: 05-04-2011
Place: 08:45-11:45
Time: CR-2M

1. Consider the nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\cos(x_1)(1-x_2)\\ x_2+\sin(2x_1)\end{bmatrix}.$$

   (a) Determine all points of equilibrium.

   (b) Determine the linearization at $\bar{x} = (\pi/2, 0)$.

2. Show that $P = \begin{bmatrix}4 & 1\\ 1 & \frac{1}{2}\end{bmatrix}$ is positive definite.

3. Determine a Lyapunov function for $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}-2 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ at equilibrium $\bar{x} = \begin{bmatrix}0\\ 0\end{bmatrix}$.

4. Formulate LaSalle's Invariance Principle.

5. Consider minimizing the cost function

$$J := \int_0^1 \tfrac{1}{2}x^2(t) + \tfrac{1}{2}(\dot{x}(t)+x(t))^2\,dt$$

   (a) Find the function $x(t)$ that satisfies the Euler equation for this cost J, with initial and final condition

$$x(0) = x_0, \qquad x(1) = 0.$$

   (b) Does this $x(t)$ satisfy the necessary second order condition of minimality of J?

6. Consider the linear system

$$\dot{x}(t) = u(t), \qquad x(0) = x_0$$

   with cost function

$$J(x_0, u) := \int_0^T \tfrac{1}{2}x^2(t) + \tfrac{1}{2}(u(t)+x(t))^2\,dt.$$

   We assume that $u(t)$ at any t is free to choose (i.e. $u(t) \in \mathbb{R}$).

   (a) Write down the Bellman equation for this problem and show that a quadratic value function of the form $V(x,t) = P(t)x^2$ will do and derive the differential equation for $P(t)$.

   (b) (deleted)g

   (c) (deleted)

(d) (deleted)

(e) (deleted)

(f) (deleted)

7. Let Ψ, Φ be two real valued functions. Suppose a twice continuously differentiable x minimizes the cost function

$$\Psi(x(T)) - \Phi(x(0)) + \int_0^T F(t, x(t), \dot{x}(t))\, dt$$

Show that this $x(t)$ satisfies the Euler equation. (Notice that this is a free initial- and end-point problem, meaning that both $x(0)$ and $x(T)$ are free to choose.)

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problem:	1	2	3	4	5	6	7
points:	3+3	2	4	3	5+2	6+3+2+2+2+3	4

Exam grade is $1 + 9p/p_{\max}$.

Euler:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Hamiltonian equations (with $H = p^T f(x,u) + L(x,u)$) for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x,p,u), \qquad x(0) = x_0,$$
$$\dot{p} = -\frac{\partial H}{\partial x}(x,p,u), \qquad p(t_e) = \frac{\partial S}{\partial x}(x(t_e))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(t_e) = G$$

Bellman:

$$\frac{\partial W}{\partial t}(x,t) + \min_{v \in \mathcal{U}}\left[\frac{\partial W}{\partial x^T}(x,t) f(x,v) + L(x,v)\right] = 0, \qquad W(x,t_e) = S(x)$$

1. (a) From the first equations we see that $x_1 = \pi/2 + k\pi$ or $x_2 = 1$. If $x_1 = \pi/2 + k\pi$ then the 2nd eqn says that $x_2 = 0$. If $x_2 = 1$ then the 2nd eqn says that $x_1 = -\pi/4 + n\pi$. So:

$$(\pi/2 + k\pi, 0) \quad \text{and} \quad (-\pi/4 + n\pi, 1)$$

(b) The linearization is $\dot{x}_\Delta = Ax_\Delta$ with

$$A := \begin{bmatrix} -\sin(x_1)(1-x_2) & -\cos(x_1) \\ 2\cos(2x_1) & 1 \end{bmatrix}\Bigg|_{x=(\pi/2,0)} = \begin{bmatrix} -1 & 0 \\ -2 & 1 \end{bmatrix}$$

2. $P_{11} = 4 > 0$ and $\det P = 4 \times \frac{1}{2} - 1 \times 1 = 2 - 1 = 1 > 0$

3. Many answers possible here.

One method: solve, $A^T P + PA = -I$ *for* P. This gives three equations in three unknowns. The solution (derivation not shown) is

$$\tilde{P} = \begin{bmatrix} 1/2 & -1/2 \\ -1/2 & 3/2 \end{bmatrix}.$$

By construction then $V(x) := x^T P x$ has time derivative $x^T(-I)x = -x_1^2 - x_2^2$ which is < 0 for all $x \neq 0$. This $\tilde{P}$ is positive definite because $P_{11} = 1/2 > 0$ and $\det \tilde{P} = 1/2 > 0$. Hence $V(x)$ is a Lyapunov function.

4. See lecture notes (either Thm. 1.2.15 or Thm. 1.2.17 whichever you like)

5. (a)

$$\begin{aligned} 0 &= \left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) \\ &= (x + (\dot{x} + x)) - \frac{d}{dt}(\dot{x} + x) \\ &= 2x + \dot{x} - \ddot{x} - \dot{x} \\ &= 2x - \ddot{x}. \end{aligned}$$

Hence

$$x(t) = \alpha \, e^{\sqrt{2}t} + \beta \, e^{-\sqrt{2}t}$$

Now

$$x_0 = x(0) = \alpha + \beta, \quad 0 = x(1) = \alpha \, e^{\sqrt{2}} + \beta \, e^{-\sqrt{2}}.$$

From the second it follows that $\alpha = -\beta \, e^{-2\sqrt{2}}$. The intitial condition now says $x_0 = \beta(1 - e^{-2\sqrt{2}})$. Hence

$$x(t) = \frac{-e^{\sqrt{2}(t-2)} + e^{-\sqrt{2}t}}{1 - e^{-2\sqrt{2}}} x_0$$

(b) $\frac{\partial^2 F}{\partial \dot{x} \partial \dot{x}} = \frac{\partial(\dot{x}+x)}{\partial \dot{x}} = 1 > 0$. It is postive, so answer is yes.

6. (a) Try $V(x,t) = x^2 P_t$ in the Belmann equations (I use subscript in t for expository reasons):

$$\begin{aligned} 0 &= \frac{\partial V}{\partial t}(x,t) + \min_{v\in\mathbb{R}} \left[\frac{\partial V}{\partial x}(x,t) f(x,v) + L(x,v) \right] \\ &= x^2 \dot{P}_t + \min_{v\in\mathbb{R}} (2xP_t v + \tfrac{1}{2}x^2 + \tfrac{1}{2}(v+x)^2) \end{aligned}$$

the minimizing v follows from differentiation: $2xP_t + (v+x) = 0$, hence $v = -x(1+2P_t)$. We continue with this v plugged in:

$$= x^2 \dot{P}_t + xP_t(-2x(1+2P_t)) + \tfrac{1}{2}x^2 + \tfrac{1}{2}(2xP_t)^2$$

As in standard LQ, a common factor x^2 can be cancelled from the Belmann equation to obtain:

$$\begin{aligned} 0 &= \dot{P}_t - 2P_t(1+2P_t) + \tfrac{1}{2} + 2P_t^2 \\ &= \dot{P}_t - 2P_t^2 - 2P_t + \tfrac{1}{2} \end{aligned}$$

and the final condition of P_t is $S(x) = 0 = x^2 P_T$, i.e., $P_T = 0$. This completes the Riccati differential equations. The solution P_t makes $V(x,t) := x^2 P_t$ satisfy the Bellman equation.

(b)

(c)

(d)

(e)

(f)

7. *Method 1 (this is a bit vague):* The Euler equation holds if we optimize over $x(t)$ with given initial and final condition. If we relax those two conditions then we optimize over a *bigger* set so the first order conditions for optimality become stronger (i.e. Euler holds *and something more*).

Method 2 (probably more convincing): Suppose $x(t)$ is an optimal solution. If Euler does not hold then a perturbation $x_\delta(t) := x(t) + \delta(t)$ with $\delta(0) = \delta(T) = 0$ exists that achieves a smaller value for $\int_0^T F(t, x_\delta(t), \dot{x}_\delta(t))\,dt$. The $\Psi(x(T)) - \Phi(x(0))$ are the same for x and x_δ because $\delta(0) = \delta(T) = 0$. So then x_δ achieves a smaller value of

$$\Psi(x(T)) - \Phi(x(0)) + \int_0^T F(t, x(t), \dot{x}(t))\,dt$$

as well.