Optimal Control
(course code: 156162)

Date: 10-04-2012
Place: CR-2M
Time: 08:45–11:45

1. Consider the nonlinear system

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}x_1^3-x_2\\ -x_1^2+x_2^2\end{bmatrix} \tag{1}$$

   (a) Determine all points of equilibrium
   (b) Determine the linearization at all points of equilibrium
   (c) Does the *nonlinear* system (1) have an unstable equilibrium?
   (d) Does the *nonlinear* system (1) have a stable equilibrium?

2. Consider minimization of

$$\int_0^1 \dot{x}^2(t)+12\,t\,x(t)\,dt,\quad x(0)=0,\quad x(1)=1$$

   over all functions $x:[0,1]\to\mathbb{R}$.

   (a) Determine the Euler equation for this problem
   (b) Solve the Euler equation

3. Under some conditions the Euler equation implies the Beltrami identity. Formulate these conditions and then derive the Beltrami identity from the Euler equation.

4. Consider the second order system with mixed initial and final conditions

$$\dot{x}_1(t)=u(t),\quad \dot{x}_2(t)=1,\qquad x_1(0)=0,\ x_2(0)=0,\ x_1(1)=1$$

   and with cost

$$J(x_0,u):=\int_0^1 u^2(t)+12x_2(t)x_1(t)\,dt$$

   The input $u:[0,1]\to\mathbb{R}$ is not restricted, i.e. $u(t)$ can take on any real value.

   (a) Determine the Hamiltonian for this problem
   (b) Determine the differential equations for state x and co-state p, including the boundary conditions
   (c) Express the candidate minimizing $u_*(t)$ as a function of $x_*(t), p_*(t)$
   (d) Solve the equations for x_*, p_*, u_* (that is, determine $x_*(t), p_*(t), u_*(t)$ as explicit functions of time $t\in[0,1]$)

5. Consider the system

$$\dot{x}(t) = x(t) + u(t), \qquad x(0) = x_0, \qquad u(t) \in \mathbb{R}$$

on the finite time horizon $t \in [0, T]$ with cost

$$J_{[0,T]}(x_0, u) = \tfrac{1}{2}x^2(T) + \int_0^T -x^2(t) - x(t)u(t)\, dt.$$

(a) Solve the Bellman equation [hint: try $V(x,t) = q(x)$]

(b) Determine all constant optimal inputs $u(t)$

(c) Determine the optimal cost

6. Under which general conditions on matrices A, B, Q, R does the solution $P(t)$ of the LQ-Riccati differential equation

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = 0$$

exist and at each t converge to a constant matrix P as $T \to \infty$? And under which additional conditions does it have the property that all eigenvalues of $A - BR^{-1}B^T P$ have strictly negative real part?

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problem:	1	2	3	4	5	6
points:	2+2+2+1	2+3	4	1+2+2+4	3+2+1	3

Exam grade is $1 + 9p/p_{\max}$.

Euler:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \qquad p(T) = \frac{\partial S}{\partial x}(x(T))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = G$$

Bellman:

$$\frac{\partial W(x,t)}{\partial t} + \min_{v \in \mathcal{U}}\left[\frac{\partial W(x,t)}{\partial x^T} f(x,v) + L(x,v)\right] = 0, \qquad W(x,T) = S(x)$$