Optimal Control
(course code: 156162)

Date: 02-07-2013
Place: HB-2E
Time: 13:45–16:45

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 1 - x_1^2/x_2^2 \\ 1 - x_1^2 x_2^2 \end{bmatrix} \tag{1}$$

   (a) Determine all points of equilibrium

   (b) Determine the linearization at all points of equilibrium

   (c) Determine the type of stability of the nonlinear system at all points of equilibrium

2. Determine a strong Lyapunov function for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   at equilibrium $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. Formulate the free end-point Euler-Lagrange theorem.

4. Let $\alpha, x_0, x_1 \in \mathbb{R}$. Consider the cost function and boundary conditions

$$J(x_0, x_1) = \int_0^1 x^2(t) - x(t)\dot{x}(t) + (t^2 + \alpha)\dot{x}^2(t)\,\mathrm{d}t, \qquad x(0) = x_0, x(1) = x_1.$$

   Suppose we found a solution $x_*(t)$ of the corresponding Euler-Lagrange equation meeting the boundary conditions. For which values of $\alpha \in \mathbb{R}$ are you sure that $x_*(t)$ minimizes J? [Note: you do not have to solve the Euler-Lagrange equation.]

5. Consider the system

$$\begin{aligned} \dot{x}_1(t) &= x_2(t) & \qquad x_1(0) &= 0, \\ \dot{x}_2(t) &= u(t) & \qquad x_2(0) &= 0 \end{aligned}$$

   with cost

$$J_{[0,1]}(u) = (x_1(2) - 10)^2 + \int_0^2 u^2(t)\,\mathrm{d}t.$$

   (a) Determine the Hamiltonian

   (b) Determine the Hamiltonian equations for the costate $p_1(t), p_2(t)$ and optimal input (expressed in terms of $p_1(t), p_2(t)$)

   (c) Determine the optimal $x_1(t), x_2(t)$ and, in particular, what is the value of $x_1(2)$?

6. Suppose

$$\dot{x}(t) = u(t), \qquad x(0) = x_0$$

and that

$$J_{[0,T]}(x_0) = \int_0^T x^4(t) + u^4(t)\,\mathrm{d}t$$

for some arbitrary positive T. Suppose $u(t)$ is free to choose in $\mathbb{R}$.

(a) Try a value function of the form $V(x,t) = x^4 h(t)$ and rewrite the resulting Hamilton-Jacobi-Bellman equations as a differential equation in $h(t)$ including a final condition on $h(T)$.

(b) Is the candidate optimal input $u(t)$ *linear* in the state (that is, of the form $u(t) = c(t)x(t)$ for some function $c(t)$ not depending on $x(t)$)?

(c) The theory of Hamilton-Jacobi-Bellman is fantastic, except for one condition: it is in general not guaranteed that the candidate optimal input $u(t)$ makes the system $\dot{x}(t) = f(x(t), u(t))$ well defined for all $t \in [0, T]$.

Is our system $\dot{x}(t) = u(t)$ well defined for our candidate optimal input? Be precise in your answer.

(d) Now take $T = \infty$. What do you think is the optimal input $u(t)$ (as a function of $x(t)$) that minimizes $J_{[0,\infty]}(x_0)$?

(e) *Prove* the claim of part (d) of this exercise. [Hint: the Hamilton-Jacobi-Bellman theorem in the lecture notes is proved for finite T only.]

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problem:	1	2	3	4	5	6
points:	2+3+3	4	3	3	2+3+5	4+2+4+2+2

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H}{\partial p}(x, p, u), \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H}{\partial x}(x, p, u), \qquad p(T) = \frac{\partial S}{\partial x}(x(T))$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = S$$

Hamilton-Jacobi-Bellman:

$$\frac{\partial V(x,t)}{\partial t} + \min_{v \in \mathbb{U}}\left[\frac{\partial V(x,t)}{\partial x^T} f(x, v) + L(x, v)\right] = 0, \qquad V(x, T) = S(x)$$