Optimal Control
(course code: 156162)

Date: 08-04-2014
Place: Sportcentrum
Time: 08:45–11:45

1. Consider the nonlinear system

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + x_1^2 - x_2^2 \\ 2x_1x_2 - 6x_2 \end{bmatrix} \tag{1}$$

   (a) Determine all points of equilibrium
   (b) Determine the linearization at all points of equilibrium
   (c) Determine the type of stability of the nonlinear system at all points of equilibrium

2. Determine a strong Lyapunov function for

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

   at equilibrium $\bar{x} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

3. Let $x_0, x_1 \in \mathbb{R}$. Consider the cost function and boundary conditions

$$J(x_0, x_1) = \int_0^T (\dot{x}(t))^2 \, \mathrm{d}t, \qquad x(0) = x_0, x(1) = x_1.$$

   (a) Determine a solution $x_*(t)$ that solve the Euler-Lagrange equation and $x_*(0) = x_0, x_*(T) = x_1$.
   (b) Is the solution of $x_*(t)$ globally optimal?

4. The classic Pontryagin's minimum principle supplies necessary conditions if the final time T is fixed. What first-order condition is added if we, now, *optimize* over the final time T as well?

5. Consider the system

$$\dot{x}(t) = 2(1 - u(t)), \qquad x(0) = 1$$

   with $u(t)$ free to choose, $u(t) \in \mathbb{R}$ and with cost

$$J_{[0,1]} = \int_0^1 -x(t) + \frac{1}{2}u^2(t) \, dt.$$

   (a) Determine the Hamiltonian
   (b) Determine the costate $p(t)$ explicitly as function of time

(c) Determine the optimal input $u(t)$ explicitly as function of time

(d) Determine the optimal state $x(t)$ explicitly as function of time

(e) Suppose next that $u(t)$ is restricted to $u(t) \in [-1, 1]$. What is the optimal input now?

6. Consider the infinite horizon cost

$$J_{[\tau,\infty)}(x(\tau), u(\cdot)) = \int_{\tau}^{\infty} L(x(t), u(t))\, dt.$$

A terminal cost is absent. We assume that the value function exists.

(a) Argue that the value function $V(x, t)$ does not depend on time t

(b) So we can write the value function as $V(x)$. Use this to simplify the HJB equation (only the differential equation; discard the final time condition.)

(c) Consider next the integrator system $\dot{x}(t) = u(t)$ and that $u(t)$ is free to choose, $u(t) \in \mathbb{R}$, and suppose that the cost is

$$J(x_0, u(\cdot)) = \int_0^{\infty} x^2(t) + u^4(t)\, dt.$$

Show that the only nonnegative solution $V(x)$ with $V(0) = 0$ of the HJB equation is $V(x) = c|x|^{\gamma}$, and determine the constant c and constant γ.

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problem:	1	2	3	4	5	6
points:	3+3+3	3	2+2	2	2+3+3+3+4	3+2+5

Exam grade is $1 + 9p/p_{\max}$.

Euler-Lagrange:

$$\left(\frac{\partial}{\partial x} - \frac{d}{dt}\frac{\partial}{\partial \dot{x}}\right) F(t, x(t), \dot{x}(t)) = 0$$

Beltrami:

$$F - \frac{\partial F}{\partial \dot{x}}\dot{x} = C$$

Standard Hamiltonian equations for initial conditioned state:

$$\dot{x} = \frac{\partial H(x, p, u),}{\partial p} \qquad x(0) = x_0,$$

$$\dot{p} = -\frac{\partial H(x, p, u),}{\partial x}, \qquad p(T) = \frac{\partial S(x(T))}{\partial x}$$

LQ Riccati differential equation:

$$\dot{P}(t) = -P(t)A - A^T P(t) + P(t)BR^{-1}B^T P(t) - Q, \qquad P(T) = S$$

Hamilton-Jacobi-Bellman:

$$\frac{\partial V(x,t)}{\partial t} + \min_{u \in \mathbb{U}} \left[\frac{\partial V(x,t)}{\partial x^T} f(x,u) + L(x,u)\right] = 0, \qquad V(x,T) = S(x)$$