Theory of Partial Differential Equations *Name:* ____________________

Final Exam (155010) *Student ID #:* ____________________

2014.01.29 *Major (opleiding):* ____________________

Exercise	*01*		*02*	*03*		*04* **or** *05*	$\sum$
	(a)	*(b)*		*(a)*	*(b)*		
Max	*5*	*4*	*9*	*4*	*4*	*10*	*36*
Grade							

Guidelines

• This is an open book exam: you **may** consult **one (1)** book—the course textbook or any other.
• You may also use your class (HC) notes, practice session (WC) notes, homework assignments and my typed-in course notes.
• Select and solve **only one** of the two problems 04 and 05; working on both means the **worst** will be graded.
• Read the statement of each problem **carefully** or you may end up solving an entirely different problem!
• Theorems and formulas in the book and/or in my typed-in notes **may be used without proof**, **unless explicitly stated otherwise**. Formulas derived in homework or exercise problems may also be used, but please write down which HW/WC problem set you copy them from.
• If in doubt about anything, please do not hesitate to **ask** the proctor overseeing the exam for a clarification.

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01. Consider the first order, quasilinear partial differential equation

$$-y\,U_x + x\,U_y = x\,U\,. \tag{1}$$

(a) Determine the characteristics of (1).
(b) Find the solution to (1) corresponding to the data $U(0,y)=1$, if it exists. (If it does not, explain why not.)

02. Solve for U the following initial–boundary value problem for the diffusion equation:

$$\begin{aligned} U_t(x,t) &= U_{xx}(x,t)\,, && \text{for all} \quad 0<x<1 \quad \text{and} \quad t>0\,,\\ U(x,0) &= \sin\left(\tfrac{\pi}{2}\,x\right)+3\,, && \text{for all} \quad 0\le x\le 1\,,\\ U(0,t) &= 1 \quad \text{and} \quad U_x(1,t)=0\,, && \text{for all} \quad t>0\,. \end{aligned} \tag{2}$$

[*Note: Derive the eigenvalues λ_n and eigenfunctions X_n explicitly; do not just copy them from your notes/the book. You may, nevertheless, assume that $\lambda_n \le 0$ for all n: that is, no positive eigenvalues exist.*]

03. Consider the following initial–boundary value problem for the wave equation on the half-line:

$$\begin{aligned} U_{tt}(x,t) &= U_{xx}(x,t)\,, && \text{for all} \quad x\ge 0 \quad \text{and} \quad t>0\,,\\ U(x,0) &= f(x)\,, && \text{for all} \quad x\ge 0\,,\\ U_t(x,0) &= 0\,, && \text{for all} \quad x\ge 0\,,\\ U(0,t) &= 0\,, && \text{for all} \quad t>0\,. \end{aligned} \tag{3}$$

(a) Solve the problem for $f(x)=\sin(x)$.
(b) Let $x_*>0$ be arbitrary but fixed. Find all functions f for which the displacement at x_* is zero *in the long term*— that is, for which there exists a time instant T such that $U(x_*,t)=0$ for all $t\ge T$.

— SELECT AND SOLVE **ONLY ONE** OF THE FOLLOWING TWO PROBLEMS —

04. Consider the following first-order problem on the plane:

$$\begin{aligned} -y\,U_x(x,y) + x\,U_y(x,y) &= g(x,y)\,, \quad \text{with} \quad (x,y) \in \mathbf{R}^2, \\ U(x,0) &= f(x)\,, \quad \text{with} \quad x \geq 0. \end{aligned} \tag{4}$$

A smooth solution $U(x,y)$ to this problem does not always exist. Explain in detail why this is so, formulate conditions for f and g which guarantee that such a smooth solution $U(x,y)$ exists and derive an explicit formula for it (possibly in another coordinate system).

[*Note: You are not asked to formulate the best possible conditions under which U exists; exercise your judgement! Also, 'explain in detail' means that, ideally, you would submit a clearly—and cleanly—written, intelligent discussion of the issue at hand with a balance between the quantitative (formulas) and the qualitative (interpretation). In plain speak: neither a list of formulas without explanation nor wordy explanations without actual mathematics.*]

05. Let Ω be the region outside the unit disk centered at the origin:

$$\Omega = \left\{(x,y)\,|\,x^2+y^2>1\right\}\ .$$

Naturally, the boundary $\partial\Omega$ is the unit circle:

$$\partial\Omega = \left\{(x,y)\,|\,x^2+y^2=1\right\}\ .$$

Additionally, let U be the unique smooth and bounded solution to the following problem:

$$\begin{aligned} U_{xx}(x,y) + U_{yy}(x,y) &= 0, \quad \text{for all} \quad (x,y) \in \Omega, \\ U(x,y) &= f(x,y), \quad \text{for all} \quad (x,y) \in \partial\Omega. \end{aligned} \tag{5}$$

Derive a formula for $U(x,y)$ by using any method you wish.

[*Hint: One way to proceed is by using our work in class and/or the book to rewrite (5) in polar coordinates (r,θ) and then working in the coordinate system $(s,\theta) = (1/r,\theta)$. If you need the expressions for $U_{xx}+U_{yy}$ in polar coordinates and/or Poisson's formula for harmonic functions on a disk, you may assume them to be known.*]

Good luck!