Exam for the course 191550105 "Theory of PDEs"
April 13, 2016, 8:45–11:45.

The use of electronic devices is not allowed. All the answers have to be motivated. Please indicate clearly if the order of your answers differs from the order of the questions or if your solution is written on different disjoint pages.

1. Consider initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant L,$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad u(L,t) = 0,$$
$$u(x,0) = f(x).$$

4pt (a) Derive a solution to this problem using the separation of variables method.

2pt (b) Specify the solution for $f(x) = \cos\frac{7\pi x}{2L}$.

6pt 2. Using the separation of variables method, solve boundary value problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant H,$$
$$u(0,y) = g(y), \; u(L,y) = 0, \qquad u(x,0) = 0, \; u(x,H) = 0.$$

Hint: first combine the product solutions into the superposition solution $u(x,y) = \sum \ldots$, then apply the inhomogeneous boundary condition.

3. Consider initial-value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

3pt (a) Let $g(x) = 1$ for all x. For which $F(x)$ and $G(x)$ is $u(x,t) = F(x-ct) + G(x+ct)$ a solution to this problem for $-\infty < x < \infty$?

3pt (b) Let $g(x) = 0$ for all x. For which $F(x)$ and $G(x)$ is $u(x,t) = F(x-ct) + G(x+ct)$ a solution to this problem for $x > 0$, $u(0,t) = h(t)$? You mau assume that u is continuous at $x = 0$ and $t = 0$.

4. For the linear differential operator $L[u] = (p(x)u'(x))' + q(x)u(x)$, $0 \leqslant x \leqslant L$, where functions $p(x) > 0$ and $q(x)$ are given, consider the following eigenvalue Sturm-Liouville problem:

$$L[\phi] + \lambda\phi = 0, \quad \frac{d\phi}{dx}(0) - \phi(0) = 0, \quad \frac{d\phi}{dx}(L) = 0.$$

2pt (a) Formulate Green's formula for this problem. How does the formula change if it is known that L is self-adjoint?

2pt (b) Is L self-adjoint?

2pt (c) Define the Rayleigh quotient for this problem.

5. For unknown $u(x,y)$, consider boundary value problem for the half-plane $x > 0$

$$\nabla^2 u = f(x,y), \quad u(0,y) = g(y).$$

3pt (a) Using the method of images, find the Green's function $G(x,y)$ for this problem. Hint: choose f be the δ function and consider homogeneous boundary conditions.

3pt (b) Assuming that the Green's function is known, provide the solution to the problem in terms of the Green's function.

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6. Consider initial-value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < \infty,$$
$$u(x,0) = f(x),$$
$$u(0,t) = g(t).$$

3pt (a) Assume $g(t) = 0$ for all $t > 0$. Which Fourier transform method, sine or cosine, would you use to solve the problem? Why? Give the solution to the problem using this method.

3pt (b) Assume $g(t) = 3$ for all $t > 0$ and $f(x) \to 3$ as $x \to \infty$. Derive a solution to this problem using a Fourier transform method. (Hint: you may either introduce $v(x,t) = u(x,t) - 3$ and first solve for $v(x,t)$, or solve directly for u using the convolution theorem.)

The grade is determined by summing up all the points earned, dividing the sum by four and adding one.