Practice exam for the course "Theory of PDEs"
April 1, 2016. Duration: 3 hours.

The use of electronic devices is not allowed. All the answers have to be motivated. Please indicate clearly if the order of your answers differs from the order of the questions or if your solution is written on different disadjoint pages.

1. Consider initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant L,$$
$$u(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0,$$
$$u(x,0) = f(x).$$

4pt (a) Derive a solution to this problem using the separation of variables method.

2pt (b) Specify the solution for $f(x) = \sin\frac{\pi x}{2L}$.

6pt 2. Using the separation of variables method, solve boundary value problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant H,$$
$$\frac{\partial u}{\partial x}(0,y) = 0, \ \frac{\partial u}{\partial x}(L,y) = 0, \qquad u(x,0) = f(x), \ u(x,H) = 0.$$

3. Consider initial-value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = g(x).$$

3pt (a) For which $F(x)$ and $G(x)$ is $u(x,t) = F(x-ct) + G(x+ct)$ a solution to this problem for $-\infty < x < \infty$?

3pt (b) For which $F(x)$ and $G(x)$ is $u(x,t) = F(x-ct) + G(x+ct)$ a solution to this problem for $x < 0$, $u(0,t) = e^{-t}$?

4. For the differential operator $L[u] = u''(x) + \lambda u(x)$, consider the following Sturm-Liouville problem:

$$L[\phi] = 0, \quad \phi(0) = 0, \quad \frac{d\phi}{dx}(1) + \phi(1) = 0.$$

2pt (a) Is L self-adjoint?

2pt (b) Define the Rayleigh quotient for this problem.

2pt (c) Provide an upper bound for the lowest eigenvalue of this problem.

5. For unknown $u(x,y)$, consider boundary value problem in the first quadrant ($x \geqslant 0$ and $y \geqslant 0$)

$$\nabla^2 u = f(x,y), \quad u(0,y) = g(y), \quad u(x,0) = h(x).$$

3pt (a) Using the method of images, find the Green function $G(x,y)$ for this problem. Hint: choose f be the δ function and consider homogeneous boundary conditions.

3pt (b) Assuming that the Green function is known, provide the solution to the problem in terms of the Green function.

6. Consider initial-value problem

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad -\infty < x < \infty,$$
$$u(x,0) = f(x).$$

2pt (a) Give the solution to the problem using the Fourier transform method. You may (but do not have to) give a derivation.

2pt (b) Find the Fourier transform of

$$f(x) = \begin{cases} x, & |x| \leqslant 1, \\ 0, & |x| > 1. \end{cases}$$

2pt (c) Use the convolution theorem to find $u(x,t)$ for the $f(x)$ given above.

The grade is determined by suming up all the points earned, dividing the sum by four and adding one.