Practice exam "Theory of PDEs"

① (a) Product solution $u(x,t) = G(t)\,\phi(x)$ gives

$$G'\phi = kG\phi''$$

$$\frac{G'}{kG} = \frac{\phi''}{\phi} = -\lambda$$ (should be positive so that solution doesn't grow in t)

b.-v. problem for $\phi(x)$:

$$\begin{cases} \phi'' + \lambda\phi = 0 \\ \phi(0) = 0,\ \phi'(L) = 0 \end{cases}$$

general solution $\phi(x) = C_1\cos\sqrt{\lambda}x + C_2\sin\sqrt{\lambda}x$

b. conditions

$$\phi(0) = C_1 + C_2\cdot 0 = 0 \Rightarrow C_1 = 0$$

$$\phi'(x) = \left(C_2\sin\sqrt{\lambda}x\right)' = C_2\sqrt{\lambda}\cos\sqrt{\lambda}x$$

$$\phi'(L) = C_2\sqrt{\lambda}\cos\sqrt{\lambda}L = 0 \Rightarrow \sqrt{\lambda}L = \frac{\pi}{2} + n\pi,\ n = 0,1,2,\dots$$

$$\sqrt{\lambda} = \left(n+\frac{1}{2}\right)\frac{\pi}{L}$$

$$\lambda = \left(n+\frac{1}{2}\right)^2\frac{\pi^2}{L^2},\quad n = 0,1,2,\dots$$

$$\phi_n(x) = C_2\sin\left(n+\frac{1}{2}\right)\frac{\pi}{L}$$

ODE for $G(t)$: $G'(t) = -\lambda k\,G(t) \Rightarrow G(t) = \underset{\text{constant}}{c}\,e^{-\lambda kt}$

Superposition solution:

$$\boxed{u(x,t) = \sum_{n=0}^{\infty} B_n \sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L}\, e^{-(n+1/2)^2(\pi/L)^2 tk}}$$

initial conditions: $u(x,0) = f(x) = \sum_{n=0}^{\infty} B_n\sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L}$

where B_n can be obtained in the standard way:

$$\boxed{B_n = \frac{2}{L}\int_0^L f(x)\sin\left(n+\frac{1}{2}\right)\frac{\pi x}{L}\,dx}$$

Answer

(b) Comparing the left and right hand sides here:

$$\left(f(x) =\right) \sin \frac{\pi x}{2L} = \sum_{n=0}^{\infty} B_n \sin\left(n+\tfrac{1}{2}\right)\frac{\pi x}{L}$$

we see that $B_n = \begin{cases} 0, & n \neq 0 \quad (n = 1,2,3, \dots) \\ 1, & n = 0 \end{cases}$

(2) The product solution $u(x,y) = h(x)\,\Phi(y)$ gives

$$-\frac{h''(x)}{h(x)} = \frac{\Phi''(y)}{\Phi(y)} = \lambda$$

(we put "$-$" sign where both b. conditions are homogeneous but it does not matter much).

Problem for $h(x)$:

$$\begin{cases} h''(x) = -\lambda h(x) \\ h'(0) = 0 \\ h'(L) = 0 \end{cases} \Rightarrow h_n(x) = \cos\frac{n\pi x}{L}$$

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2, \quad n = 0,1,2,\dots$$

Problem for $\Phi(y)$ (we neglect b.condition at $y=0$ now):

$$\begin{cases} \Phi''(y) = \lambda\Phi(y) \\ \Phi(H) = 0 \end{cases} \Rightarrow \Phi_n(y) = \sinh \overbrace{\frac{n\pi}{L}}^{=\sqrt{\lambda_n}}(y-H)$$

Thus $u(x,y) = \sum_{n=0}^{\infty} B_n \cos\frac{n\pi x}{L} \sinh\frac{n\pi}{L}(y-H)$ and

imposing b.c. at $y=0$ we get

$$u(x,0) = f(x) = \sum_{n=0}^{\infty} \underbrace{B_n \sinh\frac{n\pi(-H)}{L}}_{} \cos n\frac{\pi x}{L}$$

can be found as the coefficients of the Fourier cosine series of $f(x)$

Answer

(3) (a) We can derive (or we have learnt) that

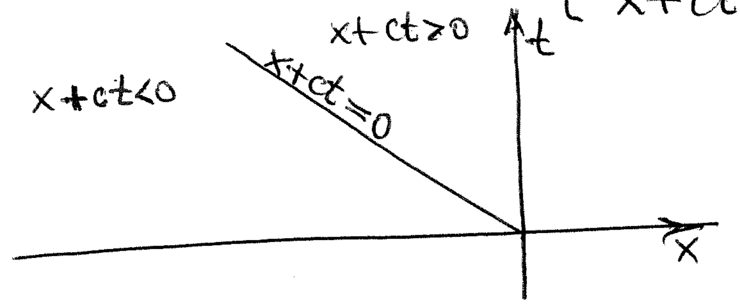
$$F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^x g(s)\,ds$$

$$G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^x g(s)\,ds \quad \text{where} \quad u(x,0) = f(x) = 0, \; u_t(x,0) = g(x)$$

Answer: $F(x) = \frac{-1}{2c}\int_0^x g(s)\,ds$

$G(x) = \frac{1}{2c}\int_0^x g(s)\,ds, \qquad F(x) = -G(x)$

(b) The formula $u(x,t) = F(x-ct) + G(x+ct)$ still holds in the same way for $\begin{cases} x-ct<0 \\ x+ct<0 \end{cases} \Leftrightarrow \begin{cases} ct>x \\ ct<-x \end{cases}$ (always true, as $x\leq 0$)



We would like to extend the formula $u = F(x-ct) + G(x+ct)$ so that it is valid for all $t \geq 0$ and $x \leq 0$.

B. conditions: $u(0,t) = e^{-t} = \underbrace{F(-ct)}_{\substack{\text{defined,} \\ \text{as } -ct<0}} + \underbrace{G(ct)}_{\substack{\text{undefined, because} \\ ct>0}} \Rightarrow$

$\Rightarrow \quad G(ct) = e^{-t} - F(-ct)$ → we now have a formula for G for positive argument;

$\boxed{G(z) = e^{-z/c} - F(-z)}$

For $x+ct<0$:

$$u(x,t) = F(x-ct) + G(x+ct) = -\frac{1}{2c}\int_0^{x-ct} g(s)\,ds + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds =$$

$$= \frac{1}{2c}\int_{x-ct}^{0} g(s)\,ds + \frac{1}{2c}\int_0^{x+ct} g(s)\,ds = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

For $x+ct \geq 0$:

$$u(x,t) = F(x-ct) + G(x+ct) = F(x-ct) + e^{-(x+ct)/c} - F(-x-ct) =$$

$$= \frac{-1}{2c}\int_0^{x-ct} g(s)\,ds + e^{-(x+ct)/c} + \frac{1}{2c}\int_0^{-x-ct} g(s)\,ds =$$

$$= \frac{1}{2c}\int_{x-ct}^{0} g(s)\,ds + e^{-(x+ct)/c} + \frac{1}{2c}\int_0^{-x-ct} g(s)\,ds =$$

$$= \frac{1}{2c}\int_{x-ct}^{-x-ct} g(s)\,ds + e^{-(x+ct)/c}$$

Answer:

$$u(x,t) = \begin{cases} \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(s)\,ds, & \text{if } x+ct<0 \\[2ex] \dfrac{1}{2c}\displaystyle\int_{x-ct}^{-x-ct} g(s)\,ds + e^{-(x+ct)/c}, & \text{if } x+ct \geq 0 \end{cases}$$

Note that in both integrals the limits of integration are negative — this is necessary, because $g(x)$ is only defined for $x<0$.

4 (a) L is self-adjoint if $\int_0^L \left(uL(v) - vL(u)\right)dx = 0$ (for all u and v satisfying b.conditions).

Using Green's formula, we have (for $L=1$)

$$\int_0^L [uL(v) - vL(u)]\,dx = \overset{=1}{p(x)}\left(uv' - vu'\right)\Big|_0^L =$$

$$= u(L)v'(L) - v(L)u'(L) - 0 = u(L)(-v(L)) - v(L)(-u(L)) = 0$$

Therefore, L is self-adjoint.

④(b) Rayleigh q. $= \dfrac{-p u u'\Big|_0^L + \int_0^L [p(u')^2 - q u^2]\,dx}{\int_0^L u^2\,dx}$

(which is formula (5.6.3) with $\sigma(x) \equiv 1$)

here $p(x) \equiv 1$, $q \equiv 0$, hence, using $u'(L) = -u(L)$, $L=1$

<u>Answer:</u> $R.q. = \dfrac{u^2\Big|_0^1 + \int_0^1 (u')^2\,dx}{\int_0^1 u^2\,dx} = \dfrac{u^2(1) + \int_0^1 (u')^2\,dx}{\int_0^1 u^2\,dx}$

④(c) To obtain a bound, we note:

$\lambda_1 \leq R.q.$ (with any u satisfying b. conditions)

Take u be piecewise linear (continuous), for instance:

$u(x) = ax, \quad x < x_*$

$u(x) = -x + b, \quad x \geq x_* \qquad 0 < x_* < 1$

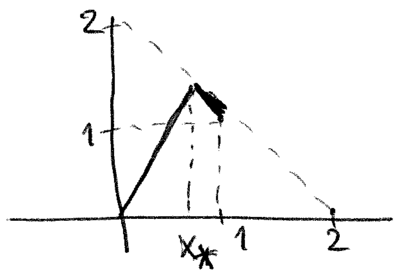
$u(1) = -1 + b = -u'(1) = 1 \quad \Rightarrow \quad b = 2$

$a x_* = -x_* + b$

$a x_* = 2 - x_*$

$(a+1) x_* = 2 \quad \Rightarrow \quad x_* = \dfrac{2}{a+1}$ take $a = 2$

Thus $u(x) = \begin{cases} 2x, & x < 2/3 \\ 2 - x, & x \geq \frac{2}{3} \end{cases}$



$u^2(1) = 1$

$\int_0^1 u^2(x)\,dx = \int_0^{2/3} (2x)^2\,dx + \int_{2/3}^1 (2-x)^2\,dx =$

$= 4\dfrac{x^3}{3}\Big|_0^{2/3} + \int_{2/3}^1 (4 + x^2 - 4x)\,dx = 4\dfrac{2^3}{3^4} + \left(4x + \dfrac{x^3}{3} - 4\dfrac{x^2}{2}\right)\Big|_{2/3}^1 = \dfrac{32}{81} + \dfrac{37}{81} =$

$= \dfrac{69}{81} \approx \dfrac{7}{8}$

4 (c) continued:

$$\int_0^1 (u')^2\,dx = \int_0^{2/3} (u')^2\,dx + \int_{2/3}^{1} (u')^2\,dx = 4\cdot\frac{2}{3} + \left(1-\frac{2}{3}\right) = 3$$

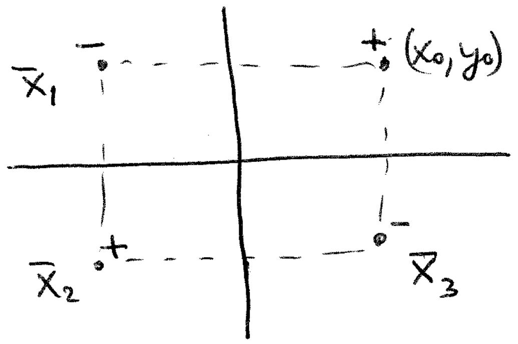
Hence $\quad R.q. = \dfrac{1+3}{7/8} = \dfrac{4\cdot 8}{7} = \dfrac{32}{7} = 4+\dfrac{4}{7} \approx 4.5$

Answer: $\lambda_1 \leq 4.5$

5 (a) For the source in $\bar{x}_0 = (x_0, y_0)$, $x_0 \geq 0$, $y_0 \geq 0$ the whole-plane Green's function is

$$G(\bar{x}, \bar{x}_0) = \frac{1}{2\pi} \ln|\bar{x} - \bar{x}_0|$$ – does not satisfy zero b. conditions on $x=0, y=0$

We add another three sources:



positive in $\bar{x}_0$ and $\bar{x}_2$
negative in $\bar{x}_1$ and $\bar{x}_3$

The ~~solution~~ Green's function is a solution of

$$\nabla^2 G = \delta(\bar{x} - \bar{x}_0) - \delta(\bar{x} - \bar{x}_1) + \delta(\bar{x} - \bar{x}_2) - \delta(\bar{x} - x_3)$$

$$G(\bar{x}, \bar{x}_0) = \sum_{k=0}^{3} (-1)^k \frac{1}{2\pi} \ln|\bar{x} - \bar{x}_k| \longleftarrow \text{answer}$$

where (x,y) are within the 1st quadrant.

(b) Green's formula: $\displaystyle \iint (u\nabla^2 G - G\nabla^2 u)\,dA = \oint (u\nabla G - G\nabla u)\cdot\vec{n}\,ds =$

$$= \int_0^\infty \Big(\underbrace{G}_{0}\frac{\partial u}{\partial y} - \underbrace{u}_{h(x)}\frac{\partial G}{\partial y}\Big)\Big|_{y=0} dx + \int_0^\infty \Big(\underbrace{G}_{0}\frac{\partial u}{\partial x} - \underbrace{u}_{g(y)}\frac{\partial G}{\partial x}\Big)\Big|_{x=0} dy$$

$$\iint u\,\delta\,dA - \iint G\,\underbrace{\nabla^2 u}_{f(x,y)}\,dA = \int_0^\infty h(x)\frac{\partial G}{\partial y}(x,0)\,dx + \int_0^\infty g(y)\frac{\partial G}{\partial x}(0,y)\,dy$$

Thus, we obtain:

$$\boxed{u(x,y) = \int_0^\infty\int_0^\infty G f(\tilde{x},\tilde{y})\, d\tilde{x}\, d\tilde{y} + \int_0^\infty h(x)\frac{\partial G}{\partial y}(x,0)\,dx + \int_0^\infty g(y)\frac{\partial G}{\partial x}(0,y)\,dy}$$

Answer

⑥ (a) After some derivation (if you like), we obtain: Fourier transform of the solution $u(x,t)$ is

$$\overline{U}(\omega,t) = \underbrace{F(\omega)}_{\substack{\text{Fourier transform}\\ \text{of the initial condition}}} e^{-k\omega^2 t}$$

and thus $u(x,t) = \int_{-\infty}^{\infty} F(\omega)\, e^{-k\omega^2 t}\, e^{-i\omega x}\, d\omega$ ← Answer.

(b)

$$F(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)e^{i\omega s}\,ds = \frac{1}{2\pi}\int_{-1}^{1} s e^{i\omega s}\,ds = \frac{1}{2\pi}\left[\frac{1}{i\omega} s e^{i\omega s}\Big|_{-1}^{1} -\right.$$

Integration by parts: $u = s,\ du = ds,\ dv = e^{i\omega s}ds,\ v = \frac{1}{i\omega}e^{i\omega s}$

$$\left. - \int_{-1}^{1}\frac{1}{i\omega}e^{i\omega s}\,ds\right] = \frac{1}{2\pi}\left[\frac{\cos\omega + i\sin\omega + \cos\omega - i\sin\omega}{i\omega} - \frac{1}{i\omega}\int_{-1}^{1} e^{i\omega s}\,ds\right] =$$

$$= \frac{2}{2\pi i\omega}\cos\omega - \frac{1}{2\pi (i\omega)^2}e^{i\omega s}\Big|_{-1}^{1} = \frac{\cos\omega}{\pi i\omega} - \frac{2i\sin\omega}{2\pi(-\omega^2)} = \frac{\cos\omega}{\pi i\omega} + \frac{i\sin\omega}{\pi\omega^2}$$

← Answer

(c) The convolution theorem: the inverse Fourier transform of the product of two Fourier transforms is $\frac{1}{2\pi}$ times the convolution of the two functions. In our case: $f(x) \to F(\omega)$ and

$$\overline{U}(\omega,t) = \underbrace{F(\omega)}_{\text{FT of } f(x)}\ \underbrace{e^{-k\omega^2 t}}_{\text{FT of } \sqrt{\frac{\pi}{kt}}\, e^{-\frac{x^2}{4kt}}}$$

Therefore: $u(x,t) = \text{Inverse FT}\left(\overline{U}(\omega,t)\right) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)\sqrt{\frac{\pi}{kt}}\, e^{-\frac{(x-s)^2}{4kt}}\,ds =$

$$= \frac{1}{2\pi}\sqrt{\frac{\pi}{kt}}\int_{-1}^{1} s\, e^{-\frac{(x-s)^2}{4kt}}\,ds$$

← Answer