Exam for the course 191550105 "Theory of PDEs"
April 19, 2017, 8:45–11:45.

The use of any electronic devices is not allowed. All the answers have to be motivated. Please indicate clearly all the questions you solve (i.e., 2(a), 2(b), ...) and avoid writing your solution at different, disjoint pages.

1. Consider initial-boundary value problem for the heat equation

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 \leqslant x \leqslant L,$$
$$\frac{\partial u}{\partial x}(0,t) = 0, \quad \frac{\partial u}{\partial x}(L,t) = 0,$$
$$u(x,0) = f(x).$$

4pt (a) Derive a solution to this problem using the separation of variables method.

2pt (b) Specify the solution for $L = 3$, $f(x) = 5 + 5\cos 3\pi x$.

6pt 2. Consider boundary value problem for the Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 \leqslant x \leqslant L, \quad 0 \leqslant y \leqslant H,$$
$$u(0,y) = 0,\ u(L,y) = 0, \qquad u(x,0) = \frac{\partial u}{\partial y}(x,0),\ u(x,H) = f(x).$$

Using the separation of variables method, solve the problem.

3. Consider initial-boundary-value problem for the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial x^2}, \quad u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x),$$
$$u(0,t) = 0, \quad u(L,t) = 0, \quad 0 \leqslant x \leqslant L.$$

3pt (a) Solve the problem by separation of the variables.

3pt (b) Assume $g(x) = 0$. For which $F(x)$ and $G(x)$ is $u(x,t) = F(x - ct) + G(x + ct)$ a solution to this problem? Show that this solution coincides with the one found in part (a).

4. For the differential operator $L[u] = u''(x) + \lambda u(x)$, consider the following Sturm-Liouville problem:

$$L[\phi] = 0, \quad \frac{d\phi}{dx}(0) = -\phi(0), \quad \phi(1) = 0.$$

3pt (a) Is L self-adjoint? Define the Rayleigh quotient for this problem.

3pt (b) Provide an upper bound for the lowest eigenvalue of this problem.

5. For unknown $u(x,y)$, consider boundary value problem for the half-plane $x < 0$

$$\nabla^2 u = f(x,y), \quad u(0,y) = g(y).$$

3pt (a) Using the method of images, find the Green's function $G(x,y)$ for this problem. Hint: choose f as the δ function and consider homogeneous boundary conditions.

3pt (b) Assuming that the Green's function is known, provide the solution to the problem in terms of the Green's function.

See the other side

6. Consider initial-boundary-value problem

$$\begin{aligned} &u_{tt} = c^2 u_{xx}, \quad x > 0, \quad t > 0, \\ &u(x,0) = 0, \quad u_t(x,0) = g(x), \\ &u(0,t) = 0. \end{aligned}$$

4pt (a) Solve the problem by applying either the sine or cosine Fourier transform method. Motivate your choice for the solution method (sine or cosine transform).

2pt (b) Using part (a) of this question, show that the solution of the sine or cosine Fourier transformed PDE is of the form $B(\omega)\sin c\omega t$. Using the odd extension of g, express $B(\omega)$ in terms of the Fourier transform $G(\omega) = \mathcal{F}[g]$.

The grade is determined by summing up all the points earned, dividing the sum by four and adding one.