Exam for the course 201700034 "Introduction to PDEs"

July 12, 2022, 8:45–11:45.

The use of any electronic devices and lecture material is not allowed. All answers must be motivated and clearly formulated. Please indicate clearly all the questions you solve (i.e., 2(a), 2(b), ...) and avoid writing your solution at different, disjoint pages.

1. Consider the following initial- boundary value problem

$$\begin{aligned} &u_{tt} = u_{xx} - xu, \quad 0 < x < 1, \quad t > 0 \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \\ &u_x(0,t) = 0, \quad u_x(1,t) + 2u(1,t) = 0. \end{aligned}$$

2pt (a) Give a physical interpretation of this problem.

2pt (b) Is the method of separation of variables applicable for the given problem? Explain your reasoning.

3pt (c) Consider the corresponding Sturm-Liouville problem:

$$L[\phi] + \lambda\phi = 0, \quad \phi'(0) = 0, \quad \phi'(1) + 2\phi(1) = 0,$$

where the differential operator $L[u] = u''(x) - xu(x)$, $x \in [0,1]$. Is L self-adjoint? Define the Rayleigh quotient for this problem.

3pt (d) Use the Rayleigh quotient to obtain a (reasonably accurate) upper bound for the lowest eigenvalue of this problem in 1(c).

2. Consider the following initial value problem for the continuously differentiable function $u(x,t)$

$$\begin{aligned} &u_t + A(u)u_x = 0, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). \end{aligned}$$

3pt (a) Assume that $A = 2u$ with the given the initial condition $u(x,0) = x/2$. Determine an explicit solution of $u(x,t)$ by using the method of characteristics.

1pt (b) Assume that $A = (1-2u)$, and $f(x) = \begin{cases} 1/4, & x < 0, \\ 1/2, & 0 < x < 1, \\ 1, & x > 1. \end{cases}$.

Give a physical interpretation of this problem.

6pt (c) Determine the solution $u(x,t)$ of the given problem in 2(b) and sketch the solution at various times.

3. Consider the following initial- boundary value problem for the heat equation

$$\begin{aligned} &u_t = u_{xx} + e^{-2t}\sin(5x), \quad 0 < x < \pi, \quad t > 0, \\ &u(0,t) = 1, \quad u(\pi,t) = 0, \\ &u(x,0) = 0. \end{aligned}$$

2pt (a) Determine the steady-state temperature distribution $u_E(x)$.

4pt (b) Find the initial boundary value problem satisfied by the transient temperature distribution $v(x,t)$, and solve this problem. (Hint: $u(x,t) = u_E(x) + v(x,t)$. The solution $v(x,t)$ must be expressed in function of the eigenvalues λ_n of the corresponding Sturm-Liouville problem.)

1pt (c) If $u(x,t)$ models the temperature of a metal rod, explain what happens with the temperature of metal rod as $t \to \infty$.

See the other side

2pt 4. (a) Show that $A(\omega)\sinh(\omega x) + B(\omega)\sinh(\omega(L-x))$ can be expressed as $P(\omega)e^{\omega x} + Q(\omega)e^{-\omega x}$, and find explicit expression for P and Q in terms of A, B, ω and L (derive the relation step by step).

1pt (b) Consider the following initial value problem and discuss what physical system is being represented.

$$u_{xx} + u_{yy} = 0, \quad 0 < x < L, \quad 0 < y < \infty,$$
$$u(x,0) = 0, \; u(0,y) = g(y), \; u(L,y) = f(y).$$

6pt (c) Solve the given problem in 4(b) by applying either the Fourier sine or cosine transform method. Motivate your choice for the solution method.

The grade is determined by summing up all the points earned, dividing the sum by four and adding one.

Table 10.5.2: Fourier Cosine Transform

$f(x) = \int_0^\infty F(\omega) \cos \omega x \, d\omega$	$C[f(x)] = F(\omega)$ $= \frac{2}{\pi} \int_0^\infty f(x) \cos \omega x \, dx$	Reference
$\frac{df}{dx}$	$-\frac{2}{\pi} f(0) + \omega S[f(x)]$	Derivatives (Sec. 10.5.4)
$\frac{d^2 f}{dx^2}$	$-\frac{2}{\pi} \frac{df}{dx}(0) - \omega^2 F(\omega)$	
$\frac{\beta}{x^2 + \beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi} \cdot \frac{\epsilon}{\epsilon^2 + \omega^2}$	Exercise 10.5.2
$e^{-\alpha x^2}$	$2 \frac{1}{\sqrt{4\pi\alpha}} e^{-\omega^2/4\alpha}$	Exercise 10.5.3
$\int_0^\infty g(\bar{x})[f(x - \bar{x}) + f(x + \bar{x})] d\bar{x}$	$F(\omega) G(\omega)$	Convolution (Exercise 10.5.7)

Table 10.5.1: Fourier Sine Transform

$f(x) = \int_0^\infty F(\omega) \sin \omega x \, d\omega$	$S[f(x)] = F(\omega)$ $= \frac{2}{\pi} \int_0^\infty f(x) \sin \omega x \, dx$	Reference
$\frac{df}{dx}$	$-\omega C[f(x)]$	Derivatives (Sec. 10.5.4)
$\frac{d^2 f}{dx^2}$	$\frac{2}{\pi} \omega f(0) - \omega^2 F(\omega)$	
$\frac{x}{x^2 + \beta^2}$	$e^{-\omega\beta}$	Exercise 10.5.1
$e^{-\epsilon x}$	$\frac{2}{\pi} \cdot \frac{\omega}{\epsilon^2 + \omega^2}$	Exercise 10.5.2
1	$\frac{2}{\pi} \cdot \frac{1}{\omega}$	Exercise 10.5.9
$\frac{1}{\pi} \int_0^\infty f(\bar{x})[g(x - \bar{x}) - g(x + \bar{x})] d\bar{x}$ $= \frac{1}{\pi} \int_0^\infty g(\bar{x})[f(x + \bar{x}) - f(\bar{x} - x)] d\bar{x}$	$S[f(x)] C[g(x)]$	Convolution (Exercise 10.5.6)

$f(x)=\int_{-\infty}^{\infty} F(\omega)e^{-i\omega x}\,d\omega$	$F(\omega)=\frac{1}{2\pi}\int_{-\infty}^{\infty} f(x)e^{i\omega x}\,dx$	Reference
$e^{-\alpha x^2}$	$\frac{1}{\sqrt{4\pi\alpha}}e^{-\omega^2/4\alpha}$	Gaussian (Sec. 10.3.3)
$\sqrt{\frac{\pi}{\beta}}e^{-x^2/4\beta}$	$e^{-\beta\omega^2}$	
$\frac{\partial f}{\partial t}$	$\frac{\partial F}{\partial t}$	Derivatives (Sec. 10.4.2)
$\frac{\partial f}{\partial x}$	$-i\omega F(\omega)$	
$\frac{\partial^2 f}{\partial x^2}$	$(-i\omega)^2 F(\omega)$	
$\frac{1}{2\pi}\int_{-\infty}^{\infty} f(\overline{x})g(x-\overline{x})d\overline{x}$	$F(\omega)G(\omega)$	Convolution (Sec. 10.4.3)
$\delta(x-x_0)$	$\frac{1}{2\pi}e^{i\omega x_0}$	Dirac delta function (Exercise 10.3.18)
$f(x-\beta)$	$e^{i\omega\beta}F(\omega)$	Shifting theorem (Exercise 10.3.5)
$xf(x)$	$-i\frac{dF}{d\omega}$	Multiplication by x (Exercise 10.3.8)
$\frac{2\alpha}{x^2+\alpha^2}$	$e^{-\lvert\omega\rvert\alpha}$	Exercise 10.3.7
$f(x)=\begin{cases} 0 & \lvert x\rvert > a \\ 1 & \lvert x\rvert < a \end{cases}$	$\frac{1}{\pi}\frac{\sin a\omega}{\omega}$	Exercise 10.3.6

Table 10.4.1: Fourier Transform

Rayleigh quotient:

$$\lambda = \frac{-p\phi\, d\phi/dx\big|_a^b + \int_a^b \left[p(d\phi/dx)^2 - q\phi^2\right] dx}{\int_a^b \phi^2\sigma\, dx},$$

Product-to-sum[30]
$2\cos\theta\cos\varphi = \cos(\theta-\varphi)+\cos(\theta+\varphi)$
$2\sin\theta\sin\varphi = \cos(\theta-\varphi)-\cos(\theta+\varphi)$
$2\sin\theta\cos\varphi = \sin(\theta+\varphi)+\sin(\theta-\varphi)$
$2\cos\theta\sin\varphi = \sin(\theta+\varphi)-\sin(\theta-\varphi)$