Exam for the course 201700034 "Introduction to PDEs"
April 19, 2023, 8:45–11:45.

The use of any electronic devices and lecture material is not allowed. All answers must be motivated and clearly formulated. Please indicate clearly all the questions you solve (i.e., 2(a), 2(b), ...) and avoid writing your solution at different, disjoint pages.

1. Consider

$$\begin{aligned} &u_{tt} = u_{xx}, \quad x > 0, \quad t > 0, \\ &u(x,0) = f(x), \quad u_t(x,0) = g(x), \quad x > 0, \\ &au_x(0,t) = bu(0,t) - h(t), \quad t > 0. \end{aligned}$$

where a and b are constants.

1pt (a) Give a physical interpretation of this problem.

1pt (b) Assume that $a = 1, b = 0$. Under which condition is the method of separation of variables generally applicable?

6pt (c) Assume that $a = 0$, $b = 1$, $f(x) = \cos(x)$, $g \equiv 0$, and $h(t) = e^{-t}$. The general solution of the PDE is given by

$$u(x,t) = F(x-t) + G(x+t)$$

Determine the solution of the initial-boundary value problem by using the general solution of the PDE.

2. Consider the following initial value problem for the continuously differentiable function $u(x,t)$

$$\begin{aligned} &u_t + Au_x = B, \quad -\infty < x < \infty, \quad t > 0, \\ &u(x,0) = f(x). \end{aligned}$$

3pt (a) Assume that $A = 3t^2$, $B = 2$ with the given the initial condition $u(x,0) = f(x)$. Determine an explicit solution of $u(x,t)$ by using the system of characteristic equations.

1pt (b) Assume that $A = 2u$, $B = 0$, and $f(x) = \begin{cases} 1, & x < 0, \\ 1+x, & 0 < x < 1, \\ 2, & x > 1. \end{cases}$.

Give a physical interpretation of this problem.

6pt (c) Determine the solution $u(x,t)$ of the given problem in 2(b) and sketch the solution at various times.

3. Consider the following initial- boundary value problem for the heat equation

$$\begin{aligned} &u_t = u_{xx} + \sin(3x), \quad 0 \leqslant x \leqslant \pi, \quad t > 0, \\ &u(0,t) = 0, \quad u(\pi,t) = 1, \\ &u(x,0) = \sin(x). \end{aligned}$$

2pt (a) Determine the steady-state temperature distribution $u_E(x)$.

4pt (b) Find the initial boundary value problem satisfied by the transient temperature distribution $v(x,t)$, and solve this problem. (Hint: $u(x,t) = u_E(x) + v(x,t)$. The solution $v(x,t)$ must be expressed in function of the eigenvalues λ_n of the corresponding Sturm-Liouville problem. Assume that all the eigenvalues λ_n are positive.)

See the other side

4pt 4. (a) Show that the Fourier cosine transform of $e^{-\alpha x^2}$ is $\frac{1}{\sqrt{\pi\alpha}}e^{-\omega^2/4\alpha}$ (derive the realtion step by step).

2pt (b) Consider the following initial value problem and discuss what physical system is being represented.

$$u_t = u_{xx} + 3u_x, \quad -\infty < x < \infty, \quad t > 0,$$
$$u(x,0) = e^{-x^2}.$$

6pt (c) Solve the given problem in 3(b) by applying the Fourier transform method. Motivate your choice for the solution method.

The grade is determined by summing up all the points earned, dividing the sum by four and adding one.