Exam: Introduction to Partial Differential Equations (201700034)

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Give a suitable explanation of your answers!
The use of electronic devices is *not* allowed. A formula sheet is not handed out.

Question 1. Introduce polar coordinates (r, ϕ) defined by $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. Assume that the function u does not depend on ϕ, i.e., $u = u(r)$. Show that

$$\Delta u = \frac{1}{r}\frac{d}{dr}\left(r\frac{du}{dr}\right).$$

Question 2. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ. Consider the classical solutions u, v of the boundary value problems

$$\begin{aligned} -\Delta u + u &= x && \text{in } \Omega, \quad \text{with } u = 0 \text{ on } \Gamma, \\ -\Delta v + v &= 1 + x && \text{in } \Omega, \quad \text{with } v = 0 \text{ on } \Gamma. \end{aligned}$$

Show that $u \leq v$.

Question 3. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and let $f \in L^2(\Omega)$. Consider the Dirichlet problem

$$-\Delta u + u = f \quad \text{in } \Omega, \text{ with } u = 0 \text{ on } \Gamma.$$

(a) Derive a variational formulation of the Dirichlet problem.

(b) Prove existence and uniqueness of a weak solution of the Dirichlet problem.

(c) Show that the weak solution of the Dirichlet problem depends continuously on the data f, i.e., derive a stability estimate.

Question 4. Let $x_i = ih$ with $h = 1/M$ and $i = 0, \ldots, M$, be a partition of the interval $(0, 1)$. Moreover, let $\Phi_i(x)$ be the associated hat functions (piecewise linear and continuous). Calculate the entries of the matrix $B \in \mathbb{R}^{M+1\times M+1}$ defined by

$$B_{i,j} = \int_0^1 \Phi_i(x)\Phi_j'(x)\,dx, \qquad i, j \in \{0, \ldots, M\}.$$

Remark: Note the derivative in the integral.

Question 5. Consider the initial-boundary value problem:

$$u_t - \Delta u + u = f \quad \text{in } \Omega \times (0, \infty), \tag{1}$$

$$u = 0 \quad \text{on } \Gamma \times (0, \infty), \tag{2}$$

$$u(\cdot, 0) = g \quad \text{in } \Omega. \tag{3}$$

(a) Let u be a classical solution of (1)–(3) with $f = 0$. Use the energy method to show that

$$\|u(t)\| \leq e^{-t}\|g\|, \quad t \geq 0.$$

Hint: You may use that, if $y'(t) + ay(t) \leq 0$ for a differentiable function $y(t) \geq 0$ and some $a > 0$, then $y(t) \leq e^{-at}y(0)$.

(b) Let u now be a classical solution of (1)–(3) for some $f \in L^2(\Omega)$, i.e., f does not depend on t, and denote $z \in H_0^1(\Omega)$ the classical solution of

$$-\Delta z + z = f \quad \text{in } \Omega, \quad \text{with } z = 0 \quad \text{on } \Gamma.$$

Show exponential convergence of u to equilibrium, i.e.,

$$\|u(t) - z\| \leq e^{-t}\|g - z\|, \quad t \geq 0.$$

Question 6. Recall the finite element discretization of the heat equation, i.e., find $u_h \in C^1([0,T]; S_h)$ such that

$$(u_h'(t), \chi) + (\nabla u_h(t), \nabla \chi) = (f(t), \chi) \quad \forall \chi \in S_h, \tag{4a}$$

$$u_h(0) = g_h. \tag{4b}$$

(a) Formulate the Euler backward scheme with time step $k > 0$ to discretize (4) in time. Denote the corresponding approximations $U^n \approx u_h(t_n)$ with $t_n = kn$.

(b) Suppose $f = 0$. Show that

$$\|U^n\| \leq \|U^{n-1}\| \qquad \text{for all } n \geq 1.$$

Question 7. Consider the initial value problem

$$u_t(x,t) + x u_x(x,t) = 0 \qquad \text{for } (x,t) \in \mathbb{R} \times (0,\infty),$$

$$u(x,0) = v(x) \quad \text{for } x \in \mathbb{R}.$$

(a) Solve the initial value problem by the methods of characteristics.

(b) Assume that $u(x,t)$ vanishes sufficiently fast for $|x| \to \infty$. Use the energy method to show that

$$\|u(\cdot,t)\| = e^{t/2}\|v\| \quad \text{for } t \geq 0.$$

Points:

Q1.		5
Q2.		4
Q3.	(a)	2
	(b)	4
	(c)	1
Q4.		6
Q5.	(a)	3
	(b)	1
Q6.	(a)	2
	(b)	3
Q7.	(a)	4
	(b)	3

Total: 38 + 2 = 40 points

Grade: (achieved points + 2)/4