Exam: Introduction to Partial Differential Equations (201700034)

P.L. Lederer and M. Schlottbom 09. July 2024, 08.45 – 11.45 2023/2024, Q-2A

Give a suitable explanation of your answers!
The use of electronic devices is *not* allowed. A formula sheet is not handed out.

Question 1. Introduce polar coordinates (r, ϕ) defined by $x_1 = r\cos\phi$, $x_2 = r\sin\phi$. Assume that the function u does not depend on r, i.e., $u = u(\phi)$. Show that

$$\Delta u = \frac{1}{r^2}\frac{d^2u}{d\phi^2}.$$

Hint: You may use that $(\arctan(\phi))' = \frac{1}{1+\phi^2}$, where $\arctan(\tan\phi) = \phi$.

Question 2. Consider the nonlinear boundary value problem

$$-u'' + u = e^{2u} \quad \text{in } \Omega = (0,1), \quad \text{with } u(0) = u(1) = 0.$$

(i) Show that all classical solutions are nonnegative, i.e., $u(x) \geq 0$ for all $x \in \bar{\Omega}$.
(ii) Show that $u(x) > 0$ for all $x \in \Omega$.

Question 3. Let $a \in \mathbb{R}$ and $a > 0$. Determine the Green's function for the boundary value problem

$$-(au')' = f \quad \text{in } (0,1), \quad \text{with } u(0) = u(1) = 0,$$

where f is a continuous function; i.e., find $G(x,y)$ such that $u(x) = \int_0^1 G(x,y)f(y)dy$, $x \in [0,1]$.

Question 4. Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with smooth boundary Γ, and let $f \in L^2(\Omega)$. Consider

$$-\Delta u + u = f \quad \text{in } \Omega, \text{ with } u = 0 \text{ on } \Gamma. \tag{1}$$

(i) Derive a variational formulation of the Dirichlet problem (1).
(ii) Let $V_h \subset H_0^1(\Omega)$ be finite-dimensional. Formulate the Galerkin approximation of (1), with solution $u_h \in V_h$.
(iii) Show that there is a constant $C > 0$ such that

$$\|u - u_h\|_{H^1(\Omega)} \leq C \inf_{v_h \in V_h} \|u - v_h\|_{H^1(\Omega)}.$$

Question 5. Consider the initial-boundary value problem:

$$u_t - \Delta u + u^3 = f \quad \text{in } \Omega \times (0,\infty), \tag{2}$$
$$u = 0 \quad \text{on } \Gamma \times (0,\infty), \tag{3}$$
$$u(\cdot, 0) = g \quad \text{in } \Omega. \tag{4}$$

(i) Let u be a classical solution of (2)–(4) with $f = 0$. Use the energy method to show that

$$\|u(t)\| \leq e^{-\lambda_1 t}\|g\|, \quad t \geq 0,$$

where λ_1 is the smallest eigenvalue of $-\Delta$ with homogeneous Dirichlet boundary conditions.

Hint: You may use that, if $y'(t) + ay(t) \leq 0$ for a differentiable function $y(t) \geq 0$ and some $a > 0$, then $y(t) \leq e^{-at}y(0)$.

(ii) Let u now be a classical solution of (2)–(4) for some $f \in L^2(\Omega)$, i.e., f does not depend on t, and denote $z \in H_0^1(\Omega)$ the classical solution of

$$-\Delta z + z^3 = f \quad \text{in } \Omega, \quad \text{with } z = 0 \quad \text{on } \Gamma.$$

Show exponential convergence of u to equilibrium, i.e.,

$$\|u(t) - z\| \leq e^{-\lambda_1 t}\|g - z\|, \quad t \geq 0,$$

Hint: You may use that $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$ and $|ab| \leq (a^2 + b^2)/2$ for $a, b \in \mathbb{R}$.

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Question 6. Recall the finite element discretization of the heat equation, i.e., find $u_h \in C^1([0,T]; S_h)$ such that

$$(u_h'(t), \chi) + (\nabla u_h(t), \nabla \chi) = (f(t), \chi) \quad \forall \chi \in S_h,$$
$$u_h(0) = g_h.$$

Consider the following time discretization, where $U^n \approx u_h(t_n)$ with $t_n = kn$, $k > 0$, and $U^0 = g_h$:

$$\frac{1}{k}(U^{n+1} - U^n, \chi) + \frac{1}{2}(\nabla(U^{n+1} + U^n), \nabla \chi) = (f(t_n + k/2), \chi) \quad \forall \chi \in S_h, \quad n \geq 0. \tag{6}$$

(i) Show that U^{n+1} is uniquely determined from (6) for all $n \geq 0$.
(ii) Show that

$$\|U^{n+1}\| \leq \|U^n\| + k\|f(t_n + k/2)\| \qquad \text{for all } n \geq 0.$$

Question 7. Consider the initial value problem

$$\frac{\partial u}{\partial t} - \begin{bmatrix} 0 & t^2 \\ t^2 & 0 \end{bmatrix} \frac{\partial u}{\partial x} = 0 \qquad x \in \mathbb{R}, t > 0, \tag{7}$$
$$u(x,0) = v(x) \quad x \in \mathbb{R}. \tag{8}$$

Solve the initial value problem (7)–(8) by the method of characteristics.

Points:

Q1. 5	**Q2.**	(i) 2	**Q3.** 6	**Q4.**	(i) 2	**Q5.**	(i) 3	**Q6.**	(i) 2	**Q7.** 5
		(ii) 2			(ii) 1		(ii) 3		(ii) 3	
					(iii) 4					

Total: 38 + 2 = 40 points **Grade:** (achieved points + 2)/4