Examination Partial Differential Equations, Course 155010, 15.11.2000, Solutions

References 'TB' refer to the textbook used in the course:
P. V. O'Neil, *Beginning Partial Differential Equations*, Wiley, New York, 1999

(1) Linear First Order PDE on a Quarter Plane

The PDE

$$\frac{1}{2}(1+x)u_x + (1+t)u_t + u = 0$$

is to be solved on the quarter plane $x > 0$, $t > 0$, subject to the condition

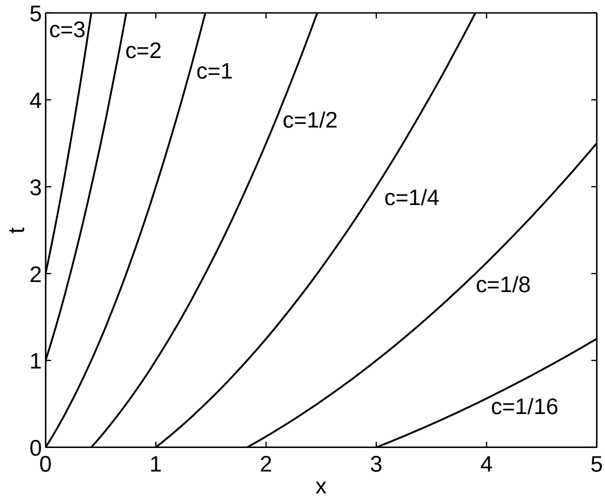
$$u(x,0) = \frac{1}{(1+x)^2} \quad \text{for} \quad x > 0\,.$$

(a) The PDE is in standard form TB(1.3) with coefficient functions $a(x,t) = (1+x)/2$, $b(x,t) = 1+t$, $c(x,t) = 1$. The characteristic equation TB(1.6) is

$$\frac{\mathrm{d}t}{\mathrm{d}x} = \frac{b(x,t)}{a(x,t)} = 2\frac{1+t}{1+x}\,.$$

If we choose a parametrization by x, determining the characteristic curves means to solve the separable ordinary differential equation $t'(x) = 2(1+t(x))/(1+x)$. Specified by a constant parameter c, the solutions are

$$t(x) = c(1+x)^2 - 1\,.$$



(b) Condition (2) prescribes u on the positive x-axis, which is not a characteristic curve. Hence (2) determines a unique solution of (1) for those points of the quarter plane D, which can be connected by a characteristic line to a point on the positive x-axis. According to the figure, these are the points below the characteristic with parameter $c = 1$, i.e. the points (x,t) with $x > 0$, $t > 0$, and $t < (1+x)^2 - 1$ (c.f. the remark in part (c)).

(c) To solve the problem, we follow the reasoning of section TB1.2. The characteristic equation induces a transformation

$$\xi = x, \qquad \eta = \frac{1+t}{(1+x)^2}$$

with nonvanishing Jacobian

$$J = \begin{vmatrix} \xi_x & \xi_t \\ \eta_x & \eta_t \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ -2\frac{1+t}{(1+x)^3} & \frac{1}{(1+x)^2} \end{vmatrix} = \frac{1}{(1+x)^2} \neq 0\,.$$

Using the relations $\xi_x = 1$, $\xi_t = 0$, $\eta_x = -2(1+t)/(1+x)^3$, $\eta_t = 1/(1+x)^2$, and $x = \xi$, $t = \eta(1+\xi)^2 - 1$ on $w(\xi(x,t), \eta(x,t)) = u(x,t)$, equation (1) can be transformed to the simpler form

$$\frac{1}{2}(1+\xi)w_\xi + w = 0\,.$$

Regarding this as a (separable) ordinary differential equation with w depending primarily on ξ, this ODE can be solved readily, where the integration constants and consequently the solution must be assumed to depend on the parameter η:

$$w(\xi,\eta) = \frac{g(\eta)}{(1+\xi)^2}\,.$$

g is a function of one variable that is to be specified later. Backtransformation to unknowns x and t yields the general solution of (1):

$$u(x,t) = \frac{1}{(1+x)^2}\,g\left(\frac{1+t}{(1+x)^2}\right).$$

Now the condition (2) demands

$$u(x,0) = \frac{1}{(1+x)^2}\,g\left(\frac{1}{(1+x)^2}\right) = \frac{1}{(1+x)^2}\,.$$

For positive x the argument of g remains bounded, $0 < 1/(1+x)^2 \leq 1$, hence (2) fixes g only in this interval:

$$g(y) = \begin{cases} 1 & \text{for } 0 < y \leq 1\,, \\ ? & \text{for } 1 < y\,. \end{cases}$$

The solution is unique only for $(1+t)/(1+x)^2 \leq 1$, i.e. for $t < (1+x)^2 - 1$, or for points (x,t) below the characteristic that runs through the origin, as was stated in (b). Nevertheless we can extend g (in any well behaved way), e.g. as $g = 1$, to obtain a solution on the entire quarter plane:

$$u(x,t) = \frac{1}{(1+x)^2}\,.$$

(2) 1-D Wave Equation

We consider the wave equation $u_{tt} = c^2 u_{xx}$ for $t > 0$ and $-\infty < x < \infty$,
with initial condition $u(x,0) = \mathrm{e}^{-x^2}$ for $-\infty < x < \infty$.

(a) A general solution of the wave equation can be written in the form

$$u(x,t) = f(x-ct) + g(x+ct)\,.$$

where f and g are two arbitray well behaved functions of one variable. Choosing

$$f(s) = g(s) = \frac{1}{2}\,\mathrm{e}^{-s^2},$$

the solution

$$u(x,t) = \frac{1}{2}\,\mathrm{e}^{-(x-ct)^2} + \frac{1}{2}\,\mathrm{e}^{-(x+ct)^2}$$

satifies the initial condition (4), and

$$u_t(x,t) = c\left\{(x-ct)\,\mathrm{e}^{-(x-ct)^2} - (x+ct)\,\mathrm{e}^{-(x+ct)^2}\right\},$$

hence $u_t(x,0) = 0$. At time t, the position of the two humps in u is given by $x \mp ct = 0$, the maxima are located at $x = \pm ct$. Thus one hump is running in positive x-direction, the other one in negative x-direction.

(b) The solution of the form $u(x,t) = g(x+ct)$ with $u_t(x,t) = c\,g'(x+ct)$ has to satisfy the initial condition (4), which leads to $g(x) = \mathrm{e}^{-x^2}$, and $u_t(x,t) = -2cx\,\mathrm{e}^{-(x+ct)^2}$. Therefore the additional initial condition (5) has to be chosen as

$$\psi(x) = u_t(x,0) = -2cx\,\mathrm{e}^{-x^2}.$$

(3) 1-D Heat Equation

The problem $u_t = u_x x$ is to be solved, for $0 < x < 1$, $t > 0$, with boundary conditions $u(0,t) = u(1,t) = 0$, $t > 0$, and initial condition $u(x,0) = \sin(3\pi x)$, $0 \le x \le 1$.

(a) Inserting the separationsansatz $u(x,t) = X(x)\,T(t)$ into the PDE leads to

$$T'\,X = T\,X'' \quad \text{or} \quad \left(\frac{X''}{X}\right)(x) = \left(\frac{T'}{T}\right)(t).$$

This requires that $X''/X = -\lambda$ and $T'/T = -\lambda$ are constants.

The initial/boundary conditions yield $u(0,t) = X(0)\,T(T) = u(1,t) = X(1)\,T(t) = 0$, thus $X(0) = X(1) = 0$, and $u(x,0) = X(x)T(0) = \sin(3\pi x)$.

The eigenvalue problem $X'' = -\lambda X$ with $X(0) = X(1) = 0$ has the solutions

$$X_n(x) = \sin(n\pi x), \quad n = 1, 2, 3, \ldots$$

with eigenvalues $\lambda_n = (n\pi)^2$. The corresponding problem $T' = -\lambda_n T$ is solved by

$$T_n(t) = \mathrm{e}^{-(n\pi)^2 t},$$

and a general solution of the heat equation, that satifies the boundary conditions, has the form

$$u(x,t) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)\,\mathrm{e}^{-(n\pi)^2 t},$$

with until now unspecified coefficients a_n.

Finally the initial condition

$$u(x,0) = \sin(3\pi x) = \sum_{n=1}^{\infty} a_n \sin(n\pi x)$$

is met, if we choose $a_3 = 1$ and $a_n = 0$ for $n \ne 3$. Thus

$$u(x,t) = \sin(3\pi x)\;\mathrm{e}^{-9\pi^2 t}$$

is the solution of the problem.

(b) For positive t,

$$\int_0^1 u(x,t)^2\,\mathrm{d}x = \mathrm{e}^{-18\pi^2 t}\int_0^1 \sin^2(3\pi x)\,\mathrm{d}x \le \int_0^1 \sin^2(3\pi x)\,\mathrm{d}x = \frac{1}{2}\left(x - \frac{1}{3\pi}\cos(3\pi x)\sin(3\pi x)\right)\Big|_0^1 = \frac{1}{2}\,.$$

(c) See theorem TB14: u achieves its maximum either on the lines $x = 0$, $t > 0$ or $x = 1$, $t > 0$, or at $t = 0$, $0 \le x \le 1$. Since $u(0,t) = u(1,t) = 0$ for $t > 0$ and $\max_{0\le x\le 1}\{u(x,0)\} = 1$, for problem (6) the weak maximum principle states that $u(x,t) \le 1$ for $0 \le x \le 1$ and $t > 0$.

(4) Laplace Equation on a Quarter Plane

An integral formula for a solution of the Laplace problem $u_{xx} + u_{yy} = 0$ on the half plane $-\infty < x < \infty$, $y > 0$ with boundary condition $u(x,0) = f(x)$ for $-\infty < x < \infty$ is

$$u(x,y) = \frac{y}{\pi}\int_{\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\,\mathrm{d}\xi\,.$$

Problem (10) has the domain restricted to a quarter plane, $u_{xx} + u_{yy} = 0$ for $x > 0$, $y > 0$, $u(x,0) = h(x)$ for $x > 0$, with an additional boundary condition on the positive y-axis:

$$\frac{\partial u}{\partial x}(0,y) = 0, \quad y > 0\,.$$

To write a solution for (10), first extend the boundary value function h to the entire x-axis

$$f(x) = \begin{cases} h(x) & \text{for } x > 0\,,\\ g(x) & \text{for } x \le 0\,,\end{cases}$$

where the extension g is to be specified later. Then, according to (9), the corresponding solution u_{hp} for the entire half plane takes the form

$$u_{\mathrm{hp}}(x,y) = \frac{y}{\pi}\int_{\infty}^{\infty} \frac{f(\xi)}{y^2 + (\xi - x)^2}\,\mathrm{d}\xi = \frac{y}{\pi}\int_{\infty}^{0} \frac{g(\xi)}{y^2 + (\xi - x)^2}\,\mathrm{d}\xi + \frac{y}{\pi}\int_{0}^{\infty} \frac{h(\xi)}{y^2 + (\xi - x)^2}\,\mathrm{d}\xi\,.$$

Upon substituting ξ by $-\xi$ in the first integral and combining the terms, this expression becomes

$$u_{\mathrm{hp}}(x,y) = \frac{y}{\pi}\int_0^{\infty}\left(\frac{g(-\xi)}{y^2 + (\xi + x)^2} + \frac{h(\xi)}{y^2 + (\xi - x)^2}\right)\mathrm{d}\xi\,.$$

To satisfy the condition on the y-axis, we are interested in

$$\frac{\partial u_{\mathrm{hp}}}{\partial x}(x,y) = -2\frac{y}{\pi}\int_0^{\infty}\left(\frac{(x+\xi)\,g(-\xi)}{(y^2 + (\xi + x)^2)^2} + \frac{(x-\xi)\,h(\xi)}{(y^2 + (\xi - x)^2)^2}\right)\mathrm{d}\xi\,,$$

and in particular in

$$\frac{\partial u_{\mathrm{hp}}}{\partial x}(0,y) = -2\frac{y}{\pi}\int_0^{\infty}\frac{\xi}{(y^2 + \xi^2)^2}\left(g(-\xi) - h(\xi)\right)\mathrm{d}\xi\,.$$

The condition $u_{\mathrm{hp},x}(0,y) = 0$ is met, if we choose g to be the even extension of h to the negative x-axis, $g(x) = h(-x)$ or $f(x) = h(|x|)$. Then u_{hp} satisfies the Laplace equation in the upper right quarter plane, and the two boundary conditions on the bordering axes. Hence u is given as the restriction of u_{hp} onto the quarter plane:

$$u(x,y) = \frac{y}{\pi}\int_0^{\infty} h(\xi)\left(\frac{1}{y^2 + (\xi + x)^2} + \frac{1}{y^2 + (\xi - x)^2}\right)\mathrm{d}\xi\,.$$