EXAM Partial Differential Equations (155010)

Wednesday, January 31, 2007, 09:00 - 12:00

1

Consider the linear first order PDE problem for a function $u = u(t, x)$

$$u_t + 2xu_x = -u, \quad -\infty < x < \infty, \quad t > 0. \tag{1}$$

a) Write down the characteristic equation of (1), and determine the characteristics. "Characteristics" are meant here as curves in the t-x - plane.

b) By applying a transformation induced by the characteristics, find a general solution of (1) that includes an arbitrary smooth function f. Show explicitly that your solution satisfies equation (1).

2

a) Classify the partial differential equation and determine the canonical form of

$$2u_{xx} - 10u_{xy} + 8u_{yy} + u_x - u_y = 0$$

b) Given the partial differential equation with Cauchy data on the y - axis

$$\begin{cases} u_{xx} + xyu_{xy} - xu_{yy} + 8u_x = 0, \\ u(0, y) = -y^2 + y, \quad u_x(0, y) = \cos(y), \end{cases} \tag{2}$$

calculate the first four terms of the Taylor expansion about $x = 0$ of the solution of (2).

3

a) Determine the Cauchy problem (wave equation + initial conditions) if the d'Alambert solution of this problem is

$$u(x, t) = \cos(3x)\cos(21t) + xt.$$

b) Assume that u satisfies an inhomogeneous 1-D wave equation on an interval $[0, L]$

$$u_{tt} = c^2 u_{xx} + f(x, t), \quad 0 < x < L, \quad t > 0,$$

with given inhomogeneity f and constant phase velocity $c > 0$. Show that the following integral relationship holds for any interval $[a, b]$ with $0 < a, b < L$:

$$\frac{d}{dt} \int_a^b \frac{1}{2}\left(u_t^2 + c^2 u_x^2\right) dx = c^2 u_t u_x \Big|_a^b + \int_a^b f u_t \, dx.$$

Hint: Multiply the wave equation by u_t, integrate over $[a, b]$, then use integration by parts on a suitable term. Note that $\partial_t(u_t^2) = 2u_t u_{tt}$.

4

Consider the initial - boundary value problem for the heat equation on a finite interval

$$u_t = 2u_{xx}, \quad \text{for} \quad 0 < x < 1, \quad t > 0$$

with homogeneous boundary conditions $u(0,t) = u(1,t) = 0$ for $t > 0$,
and initial condition $u(x,0) = f(x)$ for $0 < x < 1$.

Deduce the formal solution as found by separation of variables for this problem, and determine the solution in case the initial condition is given by $u(x,0) = \sin(3\pi x)$ for $0 < x < 1$.

5

The Laplace equation for the upper half-plane with boundary conditions on the x - axis

$$u_{xx} + u_{yy} = 0, \quad u(x,0) = f(x), \quad x \in \mathbb{R}, \quad y > 0,$$

has the solution

$$u(x,y) = \frac{1}{\pi} \int_{-\infty}^{+\infty} \frac{y f(\xi)}{y^2 + (\xi - x)^2} \, d\xi.$$

a) Check this formula in case $f(x) = k$ (constant function).

b) Now consider the equation in the left quarter plane with homogeneous boundary conditions on the negative x - axis:

$$w_{xx} + w_{yy} = 0, \quad w(x,0) = h(x), \quad w(0,y) = 0, \quad x < 0, \quad y > 0. \tag{3}$$

Find the corresponding integral expression for the dependence of the solution of (3) on the function h, given on the negative x - axis.

Grading

1 a) 2		**2 a)** 4		**3 a)** 3		**4** 7		**5 a)** 3
b) 5		**b)** 4		**b)** 4				**b)** 4

Total: $36 + 4 = 40$ points.