Theory of Partial Differential Equations *Name:* ____________________

Final Exam (155010) *Student ID #:* ____________________

2011.01.26 *Major (opleiding):* ____________________

Exercise	*01*		*02*	*03*		*04*	*05* **or** *06*	$\sum$
	(a)	*(b)*		*(a)*	*(b)*			
Max	*3*	*5*	*8*	*3*	*5*	*6*	*6*	*36*
Grade								

Guidelines

- This is an open book exam: you **may** consult **one (1)** book—the course textbook or any other.
- You may also use your class (HC) notes, practice session (WC) notes, and homework assignments.
- Select and solve **one** of the two problems 05 and 06; working on both means the **worst** will be graded.
- Read the statement of each problem **carefully** or you may end up solving an entirely different problem!
- Theorems and formulas in the book may be used without proof, **unless explicitly stated**. Please make sure you **mention which theorem/formula you are using** every time.
- If in doubt about anything, please do not hesitate to **ask** the proctor for a clarification.

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01. Consider the first order, linear partial differential equation

$$U\,U_x + U_y = 1. \tag{1}$$

(a) Determine the characteristics of (1), here meant as curves in the $(x, y, U)-$space.

(b) Is there a solution of (1) corresponding to the Cauchy data $U(x, x) = x/2$, with $-1 \le x \le 1$? If there is, determine whether it is unique. If it is unique, find it; otherwise, provide two different solutions.

02. Consider the following initial–boundary value problem for the diffusion equation:

$$\begin{aligned} &U_t = U_{xx}\,, \\ &U(0,t) = U_x(3,t) = 0\,, \quad \text{for} \quad t > 0\,, \\ &U(x,0) = \sin\left(\frac{\pi}{6}\,x\right) + 2\sin\left(\frac{5\pi}{6}\,x\right)\,, \quad \text{for} \quad 0 \le x \le 3 \quad \text{and} \quad t > 0\,. \end{aligned} \tag{2}$$

Separate variables to solve (2) (do **not** just copy the formula from the book) and calculate the maximum and minimum values of the solution over the region $[0, 3] \times [0, \infty)$ on the $(x, t)-$plane.

03. Consider the linear, second order partial differential equation

$$U_{xx} + c\,U_{xy} + 2\,U_{yy} = 0\,. \tag{3}$$

(a) Determine the type—hyperbolic, parabolic, or elliptic—of the equation, depending on the value of c. Write down the normal form of the equation for each one of these three cases and—if at all possible—write down the general solution of this normal form involving an appropriate number of arbitrary functions. (Note that you are **not** asked to determine explicitly the coordinate change that puts the equation in normal form!)

(b) For those values of c for which (3) is elliptic:

I. find new coordinates (ξ, η) which put (3) in its normal form;

II. show that this normal form does not change under translations—that is, passing to a translated coordinate system

$$\xi' = \xi + a \quad \text{and} \quad \eta' = \eta + b, \quad \text{with } a \text{ and } b \text{ (arbitrary) constants,}$$

does not alter the normal form;

III. show that the normal form does not change under rotations: passing to a rotated coordinate system

$$\xi' = \cos(\theta)\,\xi + \sin(\theta)\,\eta \quad \text{and} \quad \eta' = -\sin(\theta)\,\xi + \cos(\theta)\,\eta, \quad \text{with } \theta \text{ an (arbitrary) constant angle,}$$

does not alter the normal form.

04. Consider the 1-D wave equation on the entire spatial axis,

$$U_{tt} = c^2\, U_{xx}\,, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0\,.$$

Here, c is a positive constant. According to d'Alembert's formula, the general solution to this equation consists of a left-traveling and a right-traveling wave. Are there initial conditions which lead to right-traveling waves only? If you believe there are not, explain in detail why. If you believe there are, then find initial conditions $U(x, 0) = f(x)$ and $U_t(x, 0) = g(x)$ so that

$$U(x,t) = \begin{cases} \mathrm{e}^{-1/(x-ct)^2}, & x > ct, \\ 0, & x \le ct. \end{cases}$$

— SELECT AND SOLVE **ONLY ONE** OF THE FOLLOWING TWO PROBLEMS —

05. Consider the function U satisfying Laplace's equation in a two-dimensional bounded domain Ω and with prescribed boundary values,

$$\begin{aligned} U_{xx}(x,y) + U_{yy}(x,y) &= 0, \quad \text{where} \quad (x,y) \in \Omega, \\ U(x,y) &= f(x,y), \quad \text{for all} \quad (x,y) \in \partial\Omega. \end{aligned}$$

Then, U is the steady state (that is, time-independent) solution of the heat equation with the same boundary data:

$$\begin{aligned} V_{xx}(t,x,y) + V_{yy}(t,x,y) &= V_t(t,x,y), \quad \text{where} \quad (x,y) \in \Omega \quad \text{and} \quad t > 0, \\ V(t,x,y) &= f(x,y), \quad \text{where} \quad (x,y) \in \partial\Omega \quad \text{and} \quad t \ge 0. \end{aligned} \tag{4}$$

Both U and V satisfy a maximum principle: the maximum of U must occur on the boundary $\partial\Omega$ of Ω, whereas—for each $T > 0$—the maximum of V over $\Omega \times [0, T]$ must occur on $\mathcal{H}_T$, where

$$\mathcal{H}_T = (\Omega \times \{0\}) \cup (\partial\Omega \times [0,T]) = \{(x,y,0) : (x,y) \in \Omega\} \cup \{(x,y,t) : (x,y) \in \partial\Omega,\ 0 \le t \le T\}.$$

(Here, the symbol '$\cup$' denotes union.) Consider, now, the following **incorrect** argument which provides a way to "derive" the maximum principle for U from the maximum principle for V:

"Equip (4) with an initial condition $U(x, y, 0) = g(x, y)$, for all $(x, y) \in \Omega$. Arrange that the maximum of g lies on $\partial\Omega$, say at $(x_*, y_*) \in \partial\Omega$. Then, the maximum of V on $\mathcal{H}_T$ is $g(x_*, y_*) = f(x_*, y_*)$, and hence $V(x, y, t) < f(x_*, y_*)$. Since $t \le T$, we can let $t = T = \infty$ to find $U(x, y) = V(x, y, \infty) < f(x_*, y_*)$. This proves the maximum principle for U."

First, explain in detail where this argument fails. Then, decide whether the argument can be fixed. If it can, then fix it accordingly. If it cannot, then explain why not.

06. Consider the unique smooth solution to the following Dirichlet problem in a two-dimensional bounded domain Ω,

$$\begin{aligned} U_{xx}(x,y) + U_{yy}(x,y) &= 0, \quad \text{where} \quad (x,y) \in \Omega, \\ U(x,y) &= f(x,y), \quad \text{for all} \quad (x,y) \in \partial\Omega. \end{aligned}$$

Additionally, let W be any twice continuously differentiable function defined in $\bar{\Omega} = \Omega \cup \partial\Omega$ and satisfying the same boundary condition: $W(x, y) = f(x, y)$, for all $(x, y) \in \partial\Omega$. Assign an energy to each such function W,

$$E(W) = \frac{1}{2} \iint_{\Omega} |\nabla W|^2 \, dA, \quad \text{where} \quad \nabla W = (W_x, W_y, W_z).$$

Here, for any vector $A = (A_1, A_2, A_3)$, we write $|A|$ for its Euclidean length: $|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$. (Write $W = U + V$, for some V, and) show that U has the minimum energy among all such functions W—namely, $E(U) < E(W)$ for all $W \neq U$.

Good luck!