Theory of Partial Differential Equations Name: ______________________

Repeat Exam (155010) Student ID #: ______________________

2012.04.18 Major (opleiding): ______________________

Exercise	01		02	03		04 **or** 05	Σ
	(a)	(b)		(a)	(b)		
Max	4	6	10	4	4	8	36
Grade							

Guidelines

- This is an open book exam: you **may** consult **one (1)** book—the course textbook or any other.
- You may also use your class (HC) notes, practice session (WC) notes, and homework assignments.
- Select and solve **one** of the two problems 04 and 05; working on both means the **worst** will be graded.
- Read the statement of each problem **carefully** or you may end up solving an entirely different problem!
- Theorems and formulas in the book may be used without proof, **unless explicitly stated**. Please make sure you **mention which theorem/formula you are using** every time.
- Formulas for the **Fourier coefficients** are assumed to be **known**. You do **not** need to prove them. Your only responsibility is to use the correct formula for your Fourier coefficients.
- If in doubt about anything, please do not hesitate to **ask** the proctor for a clarification.

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01. Consider the first order, quasi-linear partial differential equation

$$U\,U_x(x,y) + U_y(x,y) = 1. \tag{1}$$

(a) Determine the characteristics of (1), here meant as curves in the $(x, y, U)-$space.

(b) Is there a solution of (1) corresponding to the Cauchy data $U(x,x) = x/2$ with $-1 \le x \le 1$? If there is, determine whether it is unique. If it is unique, find it; otherwise, provide two different solutions.

02. Consider the following initial–boundary value problem for the heat equation:

$$\begin{aligned} &U_t(x,t) = U_{xx}(x,t)\,, \quad \text{for} \quad 0 \le x \le 1 \quad \text{and} \quad t > 0\,,\\ &U_x(0,t) = U_x(1,t) = 0\,, \quad \text{for} \quad t > 0\,,\\ &U(x,0) = \sin^2(\pi\,x)\,, \quad \text{for} \quad 0 \le x \le 1\,. \end{aligned} \tag{2}$$

Separate variables to solve (2) (do **not** just copy the formula from the book) and calculate the maximum and minimum values of $U(x,t)$ over the region $[0,1] \times [0,\infty)$ on the $(x,t)-$plane. Show, also, that total heat is conserved, that is:

$$\int_0^1 U(x,t)\,dx = \int_0^1 U(x,0)\,dx\,, \quad \text{for all } t \ge 0\,. \tag{3}$$

03. Consider the 1-D wave equation on the entire spatial axis,

$$U_{tt}(x,t) = c^2\,U_{xx}(x,t)\,, \quad \text{with} \quad x \in \mathbb{R} \quad \text{and} \quad t > 0\,. \tag{4}$$

Here, c is a positive constant.

(a) Determine initial conditions $U(x,0) = f(x)$ and $U_t(x,0) = g(x)$ so that

$$U(x,t) = (x + ct)^3\,.$$

(b) The general solution $U(x,t)$ for (4) is the sum of two traveling waves: one traveling in the positive x-direction and another one traveling in the negative x-direction. Are there initial conditions such that the right-traveling wave is identically zero? That is, is it possible to find initial conditions for which the solution is a left-travelling wave?

— SELECT AND SOLVE **ONLY ONE** OF THE FOLLOWING TWO PROBLEMS —

04. Consider the Laplace equation in two dimensions,

$$U_{xx}(x, y) + U_{yy}(x, y) = 0, \quad \text{where} \quad (x, y) \in \Omega. \tag{5}$$

Here, Ω is the two-dimensional domain outside the unit disc,

$$\Omega = \left\{ (x, y) \,\middle|\, x^2 + y^2 > 1 \right\}.$$

The boundary of that domain is the unit circle,

$$\partial\Omega = \left\{ (x, y) \,\middle|\, x^2 + y^2 = 1 \right\}.$$

We write U for the unique smooth solution to (5) equipped with the boundary conditions

$$U(x, y) = f(x, y), \quad \text{for all} \quad (x, y) \in \partial\Omega.$$

Write down a formula for U. **For this problem, Poisson's formula for the solution inside the unit disc is assumed to be known.**
[Hint: Pass from Cartesian coordinates (x, y) to polar coordinates (r, θ)—you may use the formula in the book/your notes for the Laplacian expressed in polar coordinates; you do not need to prove it. Then, define new coordinates (s, θ) with θ as above and $s = 1/r$. Rewrite the problem in terms of the coordinates (s, θ) and solve it.]

05. Consider the unique smooth solution to the following Dirichlet problem in a two-dimensional bounded domain Ω,

$$\begin{aligned} U_{xx}(x, y) + U_{yy}(x, y) &= 0, \quad \text{where} \quad (x, y) \in \Omega, \\ U(x, y) &= f(x, y), \quad \text{for all} \quad (x, y) \in \partial\Omega. \end{aligned}$$

Additionally, let W be any twice continuously differentiable function defined in $\bar{\Omega} = \Omega \cup \partial\Omega$ and satisfying the same boundary condition: $W(x, y) = f(x, y)$, for all $(x, y) \in \partial\Omega$. Assign an energy to each such function W,

$$E(W) = \frac{1}{2} \iint_{\Omega} |\nabla W|^2 \, dA, \quad \text{where} \quad \nabla W = (W_x, W_y, W_z).$$

Here, for any vector $A = (A_1, A_2, A_3)$, we write $|A|$ for its Euclidean length: $|A| = \sqrt{A_1^2 + A_2^2 + A_3^2}$. (Write $W = U + V$, for some V, and) show that U has the minimum energy among all such functions W—namely, $E(U) < E(W)$ for all $W \neq U$.

Good luck!