Theory of Partial Differential Equations — *Name:* ____________________

Final Exam (155010) — *Student ID #:* ____________________

2013.01.30 — *Major (opleiding):* ____________________

Exercise	01 (a)	01 (b)	02 (a)	02 (b)	03 (a)	03 (b)	04 **or** 05	$\sum$
Max	4	4	5	3	2	8	10	36
Grade						6	~~5~~ 2	

Guidelines

- This is an open book exam: you **may** consult **one (1)** book—the course textbook or any other.
- You may also use your class (HC) notes, practice session (WC) notes, and homework assignments.
- Select and solve **only one** of the two problems 05 and 06; working on both means the **worst** will be graded.
- Read the statement of each problem **carefully** or you may end up solving an entirely different problem!
- Theorems and formulas in the book may be used without proof, **unless explicitly stated**. Please make sure you **mention which theorem/formula you are using** every time.
- **All** formulas for the **Fourier coefficients** are assumed to be **known**. You do **not** need to rederive/prove them! Your only responsibility is to select and use the **correct** formula each time.
- If in doubt about anything, please do not hesitate to **ask** the proctor overseeing the exam for a clarification.

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01. Consider the first order, quasilinear partial differential equation

$$U_x + U\,U_y = k\,U^2\,, \quad \text{where } k \text{ is a constant}. \tag{1}$$

(a) Determine the characteristics of (1), here meant as curves in the $(x, y, U)-$space.

(b) Is there a solution of (1) corresponding to the Cauchy data $U(x,0) = 1$, with $x \in (-\infty, \infty)$? If there is, determine whether it is unique. If it is unique, find it; otherwise, provide two different solutions.

02. **(a)** Solve the following initial–boundary value problem for the wave equation:

$$\begin{aligned}
&U_{tt}(x,t) = U_{xx}(x,t)\,, \quad \text{with} \quad 0 < x < \pi \quad \text{and} \quad t > 0\,,\\
&U(x,0) = \sin x\,, \quad \text{with} \quad 0 \le x \le \pi\,,\\
&U_t(x,0) = 0\,, \quad \text{with} \quad 0 \le x \le \pi\,,\\
&U(0,t) = U(\pi,t) = 0\,, \quad \text{with} \quad t > 0\,.
\end{aligned} \tag{2}$$

(b) Is there a maximum principle for the wave equation, similar to that for the heat equation and Laplace's equation? If there is, write it down. If not, explain why not.

03. Consider the linear, second order partial differential equation

$$U_{xx} + c\,U_{xy} + U_{yy} = 0\,. \tag{3}$$

(a) Determine the type—hyperbolic, parabolic, or elliptic—of the equation, depending on the value of c. Write down the normal form of the equation for each one of these three cases and—if at all possible—write down the general solution of this normal form involving an appropriate number of arbitrary functions. (Note that you are **not** asked to determine explicitly the coordinate change that puts the equation in normal form!)

(b) For the values of c making (3) elliptic, let $(\bar{x}, \bar{y}) = (\bar{x}(x,y), \bar{y}(x,y))$ be new independent variables and write

$$U(x,y) = U\left(x(\bar{x},\bar{y}), y(\bar{x},\bar{y})\right) = V(\bar{x},\bar{y})\,.$$

I. Find new coordinates $(\bar{x}, \bar{y})$ that put (3) in its normal form,

$$V_{\bar{x}\bar{x}} + V_{\bar{y}\bar{y}} = 0\,. \tag{4}$$

II. Show that this normal form does not change under rotations: pass to a rotated coordinate system

$\bar{x}' = \bar{x}\,\cos\theta + \bar{y}\,\sin\theta$ and $\bar{y}' = -\bar{x}\,\sin\theta + \bar{y}\,\cos\theta$, with θ an (arbitrary) constant angle,

and write

$$V(\bar{x},\bar{y}) = V\left(\bar{x}(\bar{x}',\bar{y}'),\bar{y}(\bar{x}',\bar{y}')\right) = W\left(\bar{x}',\bar{y}'\right) .$$

Show that W satisfies the equation

$$W_{\bar{x}'\bar{x}'} + W_{\bar{y}'\bar{y}'} = 0 .$$

III. Consider (4) again and introduce **inverted** polar coordinates

$$(\rho,\theta) = \left(\frac{1}{\sqrt{\bar{x}^2+\bar{y}^2}}\,,\ \arctan\left(\frac{\bar{y}}{\bar{x}}\right)\right) .$$

Write, also,

$$V\left(\bar{x},\bar{y}\right) = V\left(\bar{x}(\rho,\theta),\bar{y}(\rho,\theta)\right) = \Phi(\rho,\theta).$$

Derive a PDE for $\Phi(\rho,\theta)$. (*Hint*: you **may** wish to do this in two steps: (i) introduce *regular* polar coordinates (r,θ) in terms of $(\bar{x},\bar{y})$ and (ii) introduce *inverted* polar coordinates (ρ,θ) in terms of (r,θ). **You may copy the expression for the Laplacian in regular polar coordinates from your notes/book; you do not need to rederive it!**)

— SELECT AND SOLVE **ONLY ONE** OF THE FOLLOWING TWO PROBLEMS —

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04. **(a)** Consider the heat equation with no-flux boundary conditions and an arbitrary initial condition f on a two-dimensional bounded domain Ω:

$$\begin{aligned}
&U_t(x,y,t) = U_{xx}(x,y,t) + U_{yy}(x,y,t)\,, \quad \text{with} \quad (x,y)\in\Omega \quad \text{and} \quad t>0\,,\\
&U(x,y,0) = f(x,y)\,, \quad \text{with} \quad (x,y)\in\Omega\,,\\
&\hat{n}(x,y)\cdot\nabla U(x,y,t) = 0\,, \quad \text{with} \quad (x,y)\in\partial\Omega\,,\ \ t>0 \ \text{and}\ \hat{n}(x,y) \ \text{the unit vector normal to}\ \partial\Omega\ \text{at}\ (x,y).
\end{aligned} \tag{5}$$

This problem has a unique solution $U(x,y,t)$ converging to the solution of Laplace's equation on Ω:

$$\begin{aligned}
&W_{xx}(x,y) + W_{yy}(x,y) = 0\,, \quad \text{with} \quad (x,y)\in\Omega\,,\\
&\hat{n}(x,y)\cdot\nabla W(x,y) = 0\,, \quad \text{with} \quad (x,y)\in\partial\Omega\,.
\end{aligned} \tag{6}$$

(**You do not need to prove this!**) Show that, nevertheless, (6) has infinitely many solutions: if W is a solution, then so is cW for any constant c. Among these infinitely many solutions of (6), which does $U(x,y,t)$ converge to? Please substantiate your answer (also) mathematically.

05. Consider the wave equation in an interval with moving boundaries. In particular, let D be the part of the upper half-plane that is bounded by the lines $x=-t$ and $x=t$,

$$D = \{(x,t)\mid t>0 \text{ and } -t<x<t\} .$$

Then, consider the problem

$$\begin{aligned}
&U_{tt}(x,t) = U_{xx}(x,t)\,, \quad \text{with} \quad (x,t)\in D\,,\\
&U(-t,t) = f(t)\,, \quad \text{with} \quad t>0\,,\\
&U(t,t) = g(t)\,, \quad \text{with} \quad t>0\,.
\end{aligned} \tag{7}$$

Assume, also, that $f(0)=k=g(0)$, for some constant k. Find a solution $U(x,t)$ to this problem that is valid inside the domain D.

Good luck!