UNIVERSITEIT
TWENTE.

Applied Analysis Group
Dep. of Applied Mathematics
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Course: Applied Functional Analysis (191506302)

Date: Friday, April 10, 2015
Time: 8:45-11:45

- An explanation to every answer is required
- You can make use of a calculator
- (Part of) the scores to exercises 3 and 4 may be earned by homeworks, see table at the end

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1. The operator $A : L^2_{\mathbb{C}}[0,1] \to A : L^2_{\mathbb{C}}[0,1]$ is defined by

$$Af(x) = f(\sqrt{x}) \quad \text{for all } x \in \mathbb{R}\ . \qquad 0 \leq x \leq 1$$

Here $L^2_{\mathbb{C}}[0,1]$ is the complex Hilbert space of classes of square-integrable functions with inner product $\langle f, g\rangle = \int_0^1 f(t)\overline{g(t)}dt$.

(a) Show that A is linear.

(b) Determine $||A||$.

(c) Find the adjoint of A.

2. Consider the boundary-value problem for y:

$$\begin{cases} y' - y = f \\ y(0) = 0 \end{cases} \tag{1}$$

where f is a given function in the real $L^2[0,1]$.

(a) Show that $y(x) = e^x \cdot \int_0^x f(t)e^{-t}dt$ is a solution of (1) and that this solution is $Af(x)$ for some kernel operator $A : L^2[0,1] \to L^2[0,1]$.

(b) Find an upper bound for $||A||$.

3.

(a) Show that there exist *unique* real numbers a_0 and b_0 such that for all $a, b \in \mathbb{R}$ we have

$$\int_0^1 \left(|f(t) - a_0t - b_0|^2 + |f'(t) - a_0|^2\right) dt \ \leq\ \int_0^1 \left(|f(t) - at - b|^2 + |f'(t) - a|^2\right) dt$$

where $f(t)$ is a fixed continuous function on $[0,1]$.

(b) Find a_0 and b_0 in case of $f(t) = t^3$.

4. The real linear space ℓ^∞ of bounded sequences is normed by $||(a_1, a_2, a_3, \cdots)||_\infty = \sup_n |a_n|$. Define the 'Hardy' action as

$$A(a_1, a_2, a_3, \cdots) := \left(a_1, \frac{a_1 + a_2}{2}, \frac{a_1 + a_2 + a_3}{3}, \cdots\right) ,$$

so $A(\underline{a})_n = \frac{a_1+a_2+\cdots+a_n}{n}$ holds.

(a) Is A^2 a bounded or an unbounded operator?

(b) In case it is bounded, find its bound; in case it is unbounded, find its domain.

5. Suppose $K : L^2_{\mathbb{C}}[0,1] \to L^2_{\mathbb{C}}[0,1]$ is given by

$$(Kf)(t) = \int_0^1 ts(t+s)f(s)ds \qquad \text{for } 0 \le t \le 1 .$$

(a) Find all eigenvalues and eigenvectors of K.

(b) Fix $\lambda \in \mathbb{C}$. Deduce for which $g \in L^2_{\mathbb{C}}[0,1]$ the integral equation

$$f(t) - \lambda \int_0^1 ts(t+s)f(s)ds = g(t) \qquad \text{with } 0 \le t \le 1$$

has a solution $f \in L^2_{\mathbb{C}}[0,1]$.

6. The integration operator J is defined by

$$(Jf)(t) := \int_a^t f(s)ds \qquad \text{for } t \in [a,b] .$$

You may use the following information without proof:

- For each $f \in L^2(a,b)$, f is the weak derivative of Jf, i.e. $f = (Jf)'$ in the weak sense.
- $H^1(a,b) = W^{1,2}(a,b) = \{\, d\mathbf{1} + Jf \mid d \in \mathbb{R},\ f \in L^2(a,b)\}$, where $\mathbf{1}$ is the constant function with value 1.

We consider $H^1(a,b)$ as a Hilbert space with respect to its inner product

$$\langle f,g\rangle_{H^1} = \langle f,g\rangle_{L^2} + \langle f',g'\rangle_{L^2} \qquad \text{for } f,g \in H^1(a,b) .$$

(a) Find $c > 0$ such that $||Jf||_{H^1} \le c||f||_{L^2}$ for $f \in L^2(a,b)$.

(b) Show that $C^1[a,b]$ is dense in $H^1(a,b)$. ($C^1[a,b]$ is the space of all continuously differentiable functions on the interval $[a,b]$.)

Grading scheme:

1. (a) 2	**2.** (a) 3	**3.** (a) 3	**4.** 3	**5.** (a) 4	**6.** (a) 3
(b) 3	(b) 2	(b) 3		(b) 4	(b) 3
(c) 3					

Total: 36 + 4 = 40 points