UNIVERSITEIT
TWENTE.

Applied Analysis Group
Department of Applied Mathematics
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Course: Applied Functional Analysis (191506302)

Date: Tuesday, Jan 30, 2018
Time: 8:45-11:45

- An explanation to every answer is required.
- You can make use of a calculator.
- (Part of) the scores to exercises **1**(c), **3**(b) and **5**(d) may be earned by homeworks, see the underlined numbers in the table at the end of the exam.

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1. The normed vector space of all bounded real sequences is denoted by ℓ^∞, its norm by

$$||a||_\infty = \sup_{n\in\mathbb{N}} |a_n|, \text{ where } a = (a_1, a_2, a_3, \cdots) \text{ with } a_n \in \mathbb{R}.$$

(a) Check the triangle inequality for $||\cdot||_\infty$.

(b) Give an example of two vectors $a, b \in \ell^\infty$ for which

$$||a+b||_\infty^2 + ||a-b||_\infty^2 = 2\ ||a||_\infty^2 + 2\ ||b||_\infty^2$$

does not hold. What conclusion for ℓ^∞ can be drawn from such an example?

We define $c_0 := \{a \in \ell^\infty |\ \lim_{n\to\infty} a_n = 0\}$ to be the linear subspace of all sequences tending to 0.

(c) Show that c_0 is a closed subset of ℓ^∞.

2. Let $H = \ell^2$ with the standard inner product and the standard basis of unit vectors $(e_n)_{n\in\mathbb{N}}$. Define the vectors $f_n := e_n + 2e_{n+2},\ n \in \mathbb{N}$. Thus

$$f_1 = (1,0,2,0,\cdots), \quad f_2 = (0,1,0,2,0,\cdots), \quad f_3 = (0,0,1,0,2,0,\cdots), \quad \cdots$$

and let $S := \{f_n \mid n \in \mathbb{N}\}$.

(a) Find a maximal orthonormal system of $F := S^\perp$.

(b) Define $x := e_1 + e_2$ and compute the distance between x and F, so what is $\inf_{y\in F} ||x-y||$?

3. Let $\mathcal{C}^1[-1,1]$ denote the space of complex-valued, differentiable functions $f : [-1,1] \to \mathbb{R}$ with continuous derivative f' and $f(-1) = 0$. Introducing the inner product

$$\langle f, g\rangle = \int_{-1}^{1} \left(f(x)\overline{g(x)} \ + \ f'(x)\overline{g'(x)}\right)\, dx$$

we get the Sobolev space H^1 by completing $\mathcal{C}^1[-1,1]$.

(a) Prove that a Cauchy sequence (f_n) in this inner product space $\mathcal{C}^1[-1,1]$ converges uniformly to a continuous function on $[-1,1]$.

(b) Show that $\varphi : H^1 \to \mathbb{C}$ with $\varphi(f) = f(0)$ is a well-defined bounded linear map.

4. Let $A : L^2(0,1) \to L^2(0,1)$ be the operator on the complex Hilbert space $L^2(0,1)$, defined by

$$Af(x) = \int_0^x f(t)\, dt \qquad \text{for all } f \in L^2(0,1).$$

(a) Show that A does not have eigenvalues.

(b) Determine A^*.

(c) Prove that AA^* is an integral operator of the type

$$AA^*g(x) = \int_0^1 k(x,y)g(y)\, dy \qquad \text{for all } g \in L^2(0,1).$$

(d) It has been proved that the eigenvalues of AA^* are

$$\lambda_n = \frac{1}{\pi^2}\frac{4}{(2n+1)^2} \qquad \text{with } n \in \mathbb{Z}.$$

Determine the spectrum $\sigma(AA^*)$ of AA^*.

5. Let $\alpha = (\alpha_n)_{n=0}^{\infty}$ be a sequence of complex numbers. Define

$$\mathcal{D} = \left\{ x \in \ell^2(\mathbb{C}) \mid \sum_{k=0}^{\infty}(|\alpha_k x_{2k}|^2 + |\alpha_k x_{2k+1}|^2) < \infty \right\}$$

and the operator $T : \mathcal{D} \to \ell^2(\mathbb{C})$ by

$$Tx = (\alpha_0 x_1, \alpha_0 x_0, \alpha_1 x_3, \alpha_1 x_2, \cdots)$$

(a) Show that $\mathcal{D}$ is dense in $\ell^2(\mathbb{C})$.

(b) Show that T is a closed operator.

(c) Show that T is compact in case $\lim_{n\to\infty} \alpha_n = 0$.

Now, as a special case, we assume that $\alpha \in \ell^\infty(\mathbb{R})$ is given by $\alpha_0 = 0$ and $\alpha_n = \frac{1}{n}$ for $n \in \mathbb{N}$.

(d) Determine the spectral representation of T in this case.

Grading scheme:

1.	2.	3.	4.	5.
(a) 2	(a) 3	(a) 3	(a) 2	(a) 2
(b) 2	(b) 3	(b) 3	(b) 2	(b) 2
(c) 3			(c) 2	(c) 2
			(d) 2	(d) 3

Total: $36 + 4 = 40$ points