UNIVERSITY
OF TWENTE.

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Test Exam: Course *Applied Functional Analysis*

Exam time: 3h
2019-191506302-1A

Unless indicated all answers need an explanation/proof.
You may use a calculator (non-symbolic).
Notation like "$\ell^p(\mathbb{N};\mathbb{C})$" or "$L^p(0,1;\mathbb{C})$" refers to <u>*$\mathbb{C}$-valued*</u> *sequences which are p-summable or* <u>*$\mathbb{C}$-valued*</u> *'functions' which are p-integrable respectively.*
The points for each question are equally distributed over all subquestions.

NAME STUDENT: STUDENT ID:

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1. (10 points). Let X be a normed space. Define the following notions:
   (a) X is a Banach space;
   (b) the Sobolev space $H^1(0,1)$
2. (15 points). Give an example (no proof required) of
   (a) two norms on the space $\{f : [0,1] \to \mathbb{R} \colon f \text{ continuous}\}$ which are **not** equivalent;
   (b) a reflexive space
   (c) an unbounded linear operator $T : X \to \mathbb{R}$ for some normed space X over the field $\mathbb{R}$.
3. (15 points) Formulate
   (a) a version (or a corollary) of the Hahn–Banach theorem (without proof, 5p);
   (b) the spectral theorem for compact self-adjoint operators and sketch the proof (5p+5p).

4. (15 points) Consider the operator

$$A = \frac{1}{2}(J^2 + J^{*2})$$

on $L^2(a,b)$, where (as usual) $Jf(t) = \int_0^t f(x)\mathrm{d}x$ for $f \in L^2(a,b)$, $t \in (a,b)$.
Show that

   (a) A is a Hilbert-Schmidt operator and determine the integral kernel.
   (b) $\operatorname{ran} A \subset H^2(a,b)$ and $(Af)'' = f$ for all $f \in L^2(a,b)$.
   (c) Characterize the elements in $\operatorname{ran} A$ by finding the right boundary conditions.

5. (10 points) Consider the linear operator

$$S : L^1(0,1) \to L^1(0,1), (S(f))(x) = (1 + \mathrm{e}^x)f(x)$$

   (a) Show that S is bounded and determine the operator norm $\|S\|$.
   (b) Is S compact? Prove or disprove.

6. (10 points) Let $H = \ell^2(\mathbb{N})$ with canonical unit vectors $(e_n)_{n\in\mathbb{N}}$ and consider the closed linear subspaces

$$F = \overline{\operatorname{span}}\{e_{2n-1} : n \in \mathbb{N}\}, \qquad G = \overline{\operatorname{span}}\{e_{2n-1} + \tfrac{1}{n}e_{2n} : n \in \mathbb{N}\}.$$

Show that

   (a) $F + G := \{f + g \in H : f \in F, g \in G\}$ is dense in H and $F \cap G = \{0\}$.
   (b) $F + G$ is not closed.

7. (15 points)

   (a) Find $f \in L^2(0,1) = L^2(0,1;\mathbb{R})$ such that $\int_0^1 f(s)\mathrm{d}s = 1$ and such that the term

$$\int_0^1 |f(s)|^2 \mathrm{d}s$$

   is as small as possible and determine this value. Is this choice of f unique?

   (b) Find $f \in L^1(0,1) = L^1(0,1;\mathbb{R})$ such that $\int_0^1 f(s)\mathrm{d}s = 1$ and such that the term

$$\int_0^1 |f(s)| \mathrm{d}s$$

   is as small as possible and determine this value. Is this choice of f unique?

   (c) What can be said about the infimum of

$$\int_0^1 |f(s)| \mathrm{d}s$$

   over all $f \in L^1(0,1)$ such that $\int_0^1 sf(s)\mathrm{d}s = 1$? Is it a minimum?

Total points $= 4 \times 15 + 2 \times 10 + 10 = 100$