AFA EPIC CHEAT SHEET

Mooie word cover pagina

AFA is big enough as it is

We have a hundred spaces, norms on these spaces, and operators. I don't want to learn them all by heart and I don't want to read through the entire book to find the correct definitions. So I present these pages in which they can be concisely found!

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Many quick/specific definitions are dropped in this page. For definition of certain spaces continue below.

Afa Definitions

Space structure	Complete name
$(\Omega, <.,.>)$ inner pr	Hilbert
$(\Omega, \|\|.\|\|)$ Normed	Banach
$(\Omega, d(.,.))$ metric	Complete

Structural properties and THms

Inner product:

Definition 1.2. Let E be a vector space. A mapping

$$E \times E \longrightarrow \mathbb{K}, \quad (f,g) \longmapsto \langle f,g \rangle$$

is called an **inner product** or a **scalar product** if it is *sesquilinear*:

$$\langle \lambda f + \mu g, h \rangle = \lambda \langle f,h \rangle + \mu \langle g,h \rangle,$$
$$\langle h, \lambda f + \mu g \rangle = \overline{\lambda} \langle h,f \rangle + \overline{\mu} \langle h,g \rangle \qquad (f,g,h \in E,\ \lambda,\mu \in \mathbb{K}),$$

symmetric:

$$\langle f,g \rangle = \overline{\langle g,f \rangle} \qquad (f,g \in E),$$

positive:

$$\langle f,f \rangle \geq 0 \qquad (f \in E),$$

and *definite*:

$$\langle f,f \rangle = 0 \implies f = 0 \qquad (f \in E).$$

Norm:

Definition 2.5. Let E be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. A mapping

$$\|\cdot\| : E \longrightarrow \mathbb{R}_+$$

is called a **norm** on E if it has the following properties:

1) $\|f\| = 0 \iff f = 0$ for all $f \in E$ (definiteness)
2) $\|\lambda f\| = |\lambda| \|f\|$ for all $f \in E$, $\lambda \in \mathbb{K}$ (homogeneity)
3) $\|f+g\| \leq \|f\| + \|g\|$ for all $f,g \in E$ (triangle inequality

Operator definitions, for T

Bounded $\exists c \geq 0 \ s.t. \ \|Tf\| \leq c\|f\| \ \forall f \in E.$

Operator norm $\|T\| := sup_{\|f\| \leq 1} \|Tf\|$

Finite $\boldsymbol{rank(T)}$ $:= \dim(range(T)) < \infty$

Finitely approximable if $\exists (T_n) \in \mathcal{L}(E;F)$ all of finite rank such that $\|T_n - T\| \to 0.$

Compact operator (f_n) $bounded\ in\ E \to (Tf_n) \subseteq F$ has a convergent subsequence.

Definitions

CENTRAL: Complete: every Cauchy sequences converges to an element in the space.

$\boldsymbol{(f_n)}$ **is Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ n,m > N \to \|f_n - f_m\| < \epsilon.$

Open set if $O: \forall x \in O\ \exists \epsilon > 0: B(x,\epsilon) \subseteq O.$

Closed set $A = \bar{A} \subseteq \Omega$
if $x_n \in A\ \ s.t.\ x_n \to x \in \Omega \Rightarrow x_n \to x \in A.$

Closure $\bar{\boldsymbol{A}} := \{x \in \Omega \mid \exists x_n \in A\ \ s.t.\ x_n \to x\}$

Compact $\boldsymbol{(\Omega, d)}$ is when every sequence in the set has a convergent subsequence.

Compact support: when a function f vanishes outside a finite interval (a,b).

Convex: $f, g \in A, t \in [0,1]$
$\to tf + (1-t)g \in A$

A is Lebesque Measure is $\lambda^*(A) := \inf \Sigma |Q_n|$ with Q_n a sequence of intervals that cover A. If the Lebesque measure is finite, the set is **Lebesque measurable.**

$\boldsymbol{f \in A}$ **is Lebesque measurable** if $\{t \in A \mid a \leq f(t) \leq b\} \in \Sigma\ \forall a,b \in \mathbb{R}.$

Isometry: $\|Tf\|_F = \|f\|_E\ \forall f.$
Always 1-1, if also onto, then **isometric isomorphism.**

A normed space is **seperable** if $\exists A \subseteq E$ countable s.t. $\overline{span}(A) = E.$
e.g. $l^2(\mathbb{N}) is\ seperable\ with\ A = \{e_n\}.$
$l^\infty\ is\ not.$

Spaces of note

For below: I made this to be read left to right, to highlight symmetries. Does not mean *everything* placed next to each other has symmetry, but you will see the places where it obviously does have symmetry.

Norms: D_p = discrete p-norm $(\Sigma|x_n|^p)^{1/p}$, C_p = integral p-norm, $\left(\int |f|^p ds\right)^{1/p}$. OP $= \|T\| = \inf_{\|f\|\leq 1} \|Tf\|$

Space	Norm		Space	Norm	
$c := \{(x_n) \subseteq \mathbb{K} \mid x_n \to x \in \mathbb{K}\}$	$D_{p=1,p,\infty}$	Y Y Y	$C(E;F) := \{A \in \mathcal{L}(E;F) \mid A\ is\ a\ compact\ operator\}$	OP	
$c_0 := \{(x_n) \mid x_n \to 0\}$			$C_0(E;F) := \{A \in \mathcal{L}(E;F) \mid A\ is\ finitely\ approximable\}$	OP	
$c_{00} := \{(x_n) \mid x_n = 0, eventually\}$	$D_{p=1,p,\infty}$	N N N	$C_{00}(E;F) := \{A \in \mathcal{L}(E;F) \mid A\ has\ finite\ rank\}$	OP	
$\mathcal{B}(\Omega) :=$ *all bounded functions on* Ω Note not necessarily continuous. However, this kind of proves completeness for the ones below, due to closedness of them.	C_1 C_p C_∞	N N Y			
$C :=$ the set of continuous functions $C^1 :=$ the set of once cont. diff. functions. $C_b := \{f \in \mathcal{B}(\Omega) \mid f\ is\ continuous\}$ $C_0 := \{f\ cont \mid f(a) = f(b) = 0\}$ Or continuous functions with compact support. $C_{per} := \{f\ cont \mid f(0) = f(1)\}$	C_1 C_p C_∞	N N Y			
$C_0^1 := C_0 \cap C^1$	C_1 C_p C_∞	N N N	(For below: let X be any interval, and) Σ ($\subseteq P(\mathbb{R})$): =the set of all Lebesque-measurable sets Is the set on which we restrict ourselves.	---	---
			$M(X) := \{f: X \to \mathbb{K} \mid f\ is\ Lebesque - Measurable\}$	---	---
			$\mathcal{L}^1 := \{f \in M(X) : \|f\|_1 < \infty\}$	---	---
$l_1 := \{(x_n) \mid \Sigma|x_n| < \infty\}$	D_1	Y	$L_1 := \mathcal{L}^1/\sim\lambda$ with equivalence relation for almost everywhere	C_1	Y
$l_2 := \{(x_n) \mid (\Sigma|x_n|^2)^{1/2} < \infty\}$	D_2	Y	$L_2 :=$ “”	C_2	Y
$l_p := \{(x_n) \mid (\Sigma|x_n|^p)^{1/p} < \infty\}$	D_p	Y	$L_p :=$ “”	C_p	Y
$l_\infty := \{(x_n) \mid \sup|x_n| < \infty\}$	D_∞	Y	$L_\infty :=$“” with $\|f\|_{L^\infty} := \inf\{c \geq 0 \mid |f| \leq c,\ a.e.\}$	$C\infty$	Y
			$St([a,b];E) := \{f: [a,b] \to E \mid \exists\ partitioning\ (t_n)\ of\ [a,b]$ $s.t.\ \ f(t) = x_n\ on\ [t_{n-1}, t_n]\}$	C_∞	N
			$Reg([a,b];E) := \overline{St([a,b];E)}$ in $\mathcal{B}([a,b];E)$ w.r.t $\|.\|_\infty$	C_∞	Y
$\mathcal{L}(E;F)$:=the space of bounded linear functions from E to F.	OP, when F is Banach	Y	$E' := \mathcal{L}(E;\mathbb{K})$ the dual space to normed space E	OP by left side	Y
$H^1 := \{f \in L^2 \mid \exists a\ weak\ derivative\ f'\}$ $< f,g >_{H^1} = < f,g >_{L^2} + < f',g' >_{L^2}$	$\|f\|_{H^1} = \left(\|f\|_2^2 + \|f'\|_2^2\right)^{1/2}$	Y	$L_c^p := \{f \in L^p,\ \ f\ has\ compact\ support\}$	C_p	Y
$H_0^1 := H^1 \cap C_0$ $< f,g >_{H_0^1} = < f',g' >_{L^2}$	$\|f\|_{H^1}$ OR $\|f\|_{H_0^1} = \|f'\|_2$	Y Y			
$H^p := \{f \in H^1 \mid f' \in H^{p-1}\}$ $< f,g >_{H^p} = \Sigma_{k=0}^p < f^{(k)}, g^{(k)} >_{L^2}$	$\|f\|_{H^p}^2 = \Sigma \left\|f^{(k)}\right\|_2^2$	Y			
$H_0^2 =: dom(\Delta_D) = dom(L)$	Same as above for p=2 Note: closed subspace of $H^2(a,b)$, therefore→	Y			

Baire theorem consequences:

THM: A normed space with a countable algebraic basis is never complete. Note that **countable** implies infinite elements. So in this context, R^3 is complete, since it has a finite, nut a countable, basis.

On scales, strong and weak norms

$l^1 \subseteq l^2 \subseteq l^\infty$ In fact all of these are **strict**. (ex 3.4)

On finite dimensional (linear) spaces, all norms are equivalent (because they are all equivalent to the euclidean 2-norm on K^d, and there is an isometric isomorphism from E to K^d).

$L^\infty(a,b) \subseteq L^p(a,b) \subseteq L^1(a,b)$. All of these are proper inclusions.
None of these hold if we replace intervals with R.

Furthermore for $\frac{1}{q}+\frac{1}{p}=1$, $||f||_1 \leq (b-a)^{\frac{1}{q}}||f||^p$ and $||f||_p \leq (b-a)^{\frac{1}{p}}||f||_\infty$

The whole finite-approximation of operator – spaces:
$\mathcal{C}_{00} \subseteq \mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{L}(E;F)$.

Strong / weak norms

A useful tool to determine densities of spaces in each other wrt certain norms is the idea of a strong vs weak norm, since a space being dense in another wrt a strong norm is also dense wrt a weaker norm.

Def a norm is strong compared to weak when $||.||_s \leq c||.||_w$ for some c.

In the below, the constant is omitted.

$$||f||_1 \leq ||f||_p \leq ||f||_\infty$$

$$\left\|A_{[k]}\right\|_{\mathcal{L}} \leq \left\|A_{[k]}\right\|_{HS} \, the\ Hilbert-Schmidt\ norm\ for\ integral\ operators\ (OP\ theory)$$

Densities

Note that A is dense in Ω if $\bar{A} = \Omega$, with $\bar{A}$ $the\ closure\ of\ A := \{x \in \Omega: \exists (x_n) \to x, x_n \in A\}$

Before we dive in, some useful density theorems:

Approximation theorems

TH. Dense in dense = dense: $A, B \subseteq (\Omega, \mathrm{d})\ with\ A \subseteq \bar{B}\ with\ \bar{A} = \Omega, then\ \bar{B} = \Omega$

TH. Dense = dense in a weaker norm:

TH. Strong vs weak norms: $On\ (\Omega, ||.||_s)\ we\ have\ ||f - f_n||_s \to 0,\ then\ ||f - f_n||_w \to 0, too$

Cor: $A\ dense\ in\ \Omega\ wrt\ ||.||_s \to A\ dense\ in\ \Omega\ wrt\ ||.||_w$

TH. Image of dense is dense in the image: $T: E \to F\ linear, A \subseteq E\ dense, then\ T(A)\ dense\ in\ T(E)$

Density table:

Space 1	Is dense in space 2	Wrt norm (strongest)
c_{00}	l^2	2
c_{00}	c_0	inf
c_{00}	l^p	p
Weierstrass $P[a, b]$	$C[a, b]$	sup
Cor C^∞	C	sup
C_0	C	2
$C_0[a, b]$	$C[a, b]$	p
$C[a, b]$	$L^p(a, b)$	p
C_0^1	C_0	Sup/inf
$C_0^1[a, b]$	$L^p(a, b)$	p
$PL[a, b]$ piecewise linear	$C[a, b]$	inf
$L_c^p(R)$ compact support	$L^p(R)$	p (<inf)
$C_c^\infty(a, b)$	$C_0[a, b]$	Inf
$C_c^\infty(a, b)$	$L^p(a, b)$	p
$C_c^\infty(R)$	$L^p(R)$	p
$L_c^p(R)$	$L^p(R)$	p
Weierstrass 2.0 $span\{e^{2\pi ins}\}$ trigonometric polynomials	$C_{per}[0,1]$	inf
$C^1[a, b]$	$H^1(a, b)$	H-1
$C_0^1[a, b]$	$H_0^1(a, b)$	H-01

Note: the "wrt norm" column might be abundant, since, given a normed space $(E, ||.||_E)$, if a space is dense in this larger space, it will always be with respect to the norm $||.||_E$. Only when spaces allow for several norms, it is important, but usually it will be obvious.

Operators

Note on spaces such as L^p we naturally pair it with the p-norm.

Operator name	Space to space	Definition	Bounded/operator norm?
Projection	E->E inner product spaces Strictly speaking maps to F a subspace of E	$Pf := \sum < f, e_j > e_j$ With $(e_j) \in E$ an orthonormal system	$\|\|Pf\|\| \leq \|\|f\|\|$
Any operator	$K^d \to F$ From the fields with standard Euclidean norm to any normed space on K^d	…	Yes, CH 2.
Any operator	$F \to E$ with E fin dim	…	Yes, isom-isom K^d->E + above
Shift operators,	On $l^p \to l^p$	$(Lf)(n) = f(n+1)$, $(Rf)(n) = f(n-1)$, Where left deletes the first entry and right adds a 0.	Both with norm 1.
Multiplication operator	Specifically $l^2 \to l^2$	Given $(\lambda_n) \in l^\infty$, $(A_\lambda f) = (\lambda_n f(n))$	Op norm is $\|\|\lambda\|\|_\infty$
Multiplication continuous	$T_m: C \to K$	Given $m \in C$ $T_m f = \int m(s) f(s)$	Op norm is $\|\|m\|\|_1$
	$T_m: C \to (C, \infty)$	$Af = mf$	$\|\|m\|\|_\infty$
Integrator	$J: L^1(a,b) \to (C[a,b], \infty)$	$Jf(t) := \int_a^b \mathbf{1}_{[a,t]} f d\lambda$ $= \int_a^t f(x) dx$	1, $\|\|Jf\|\|_\infty < \|\|f\|\|_1$
Laplace	$\mathcal{L}: L^1(R_+) \to L^\infty(K)$	$(\mathcal{L}f)(t) := \int_{0\to\infty} e^{-ts} f(s)\, ds$	1
Fourier	$\mathcal{F}: L^1(R) \to L^\infty(R)$	$\mathcal{F}f(t) := \int_{-\infty}^{\infty} e^{-its} f(s) ds$	1
Orthogonal projection	$P_F: H \to F$ H Hilbert, F a closed subspace	$*Pf := \sum_j < f, e_j > e_j$ When $\exists (e_j)\ ONS\ in\ H$ $s.t.\ \ F := \overline{span}\{e_j\}$	1. Also: ~, **Parseval:** $\|\|Pf\|\|^2 = \sum_j \| < f, e_j > \|^2$
Derivative	$H^1(a,b) \to L^2(a,b)$	$f \to f'$	Yes
Dirichlet-Laplacian	$\Delta_D: H_0^2(a,b) \to L^2(a,b)$, As well as Δ_D^{-1}.	$\Delta_D u := u''$	Yes

Operator theory

Def Integral operator

$$For\ X, Y\ intervals\ on\ \mathbb{R}. An\ operator\ A\ is\ an\ \boldsymbol{integral\ operator}\ if\ \exists function$$

$k: X\ x\ Y \to \mathbb{K}$ such that $(Af)(t) = \left(A_{[k]}f\right)(t) := \int_Y k(t,s)f(s)ds.$

Where we call k the **kernel** of the operator.

Furthermore we define the following cross operator:

$$(f \otimes g)(x,y) = f(x)g(y)$$

Where, if f and g are measurable, so is their product. This induces a norm on L(X x Y) as you would expect.

Def $T: E \to F\ is\ invertible\ if\ T\ is\ bijective\ and\ T^{-1} is\ bounded.$

Def Hilbert-Schmidt kernel functions:
For X, Y and k as before, with $k \in L^2(X\ x\ Y)$, i.e., $\int_X \int_Y |k(x,y)|^2 dydx < \infty$,
We call k a Hilbert-Schmidt kernel-function.

Theorem then the induced HS-integral operator $A_{[k]}$ satisfies

$$\left\|A_{[k]}f\right\|_{L_2} \leq \|k\|_{2(X\ x\ Y)} \|f\|_{2(Y)}$$

And, since k is in essence bounded as a HS kernel, we have that the integral operator is bounded.

Moreover: $k\ is\ uniquely\ determined\ by\ A_{[k]}$ (a.e.).

Def HS-norm: $\left\|A_{[k]}\right\|_{HS} := \|k\|_{2(X\ x\ Y)}$

It basically takes the norm of the kernel to define the norm of the corresponding integral operator.

Approximations of operators

From the fact that $F\ Banach \to \mathcal{L}(E;F) is\ Banach$, it follows that $\|ST\| \leq \|S\|\|T\|$.

This allows for **def Strong convergence is when** $(T_n f) \to Tf\ \ \forall f \in F, in\ \|.\|_F$.

Note that $\|T_n f - Tf\| \leq \|T_n - T\|\|f\|$, hence convergence in the operator norm implies strong convergence. So in fact, strong convergence is weaker than convergence in the operator norm.

It is in fact strictly weaker: the projection does converge strongly, $P_n := \sum_{j=1}^n <.,e_j> e_j$ has

$P_n f \to f$ for each $f \in H$. However, the operators never converge in the operator norm.

Defnitions we call $A: E \to F$:

Name	Corresponding space	Definition
Finite rank	C_{00}	$\dim range(A) < \infty$
Finitely approximable	C_0	$\exists(A_n)$ all finite rank: $\|A - A_n\| \to 0$
Compact	C	$(f_n)\ bnd\ in\ E \to (Af_n)$ has a convergent subsequence.

Examples/theorems:
- HS integral operators are finitely approximable.
- $E, F\ Banach\ \ and\ \ A: E \to F$ is finitely approximable $\to A\ is\ compact$.
- $A \in \mathcal{C}_0 \to AC, DA \in \mathcal{C}_0$ for C,D just linear operators.
- $E, F\ Hilbert \to \mathcal{C}_0(E; F) = \mathcal{C}(E; F)$.

Adjoints

On Hilbert spaces, $A^*: F \to E\ adjoint\ to\ A: E \to F\ bnd\ linear,$

Is such that $< Af, g > = < f, A^*g >$.

Construction of A*: $let\ b : H\ x\ K \to \mathbb{K}\ bnd. Then\ \forall f \in H, g \in K, we\ have\ that$

$|b(f,g)| \leq c\|f\|\|g\|$, for some c. $Then\ \exists!\ lin. bnd. B: K \to H\ s.t.\ \ b(f,g) = < f, Bg >_H\ \ \forall f, g.$

Even: $\|B\| \leq c$. Then we can just take $b(f,g) = < Af, g >$, and by using Riez-Fréchet:

$$A^* = B, c = \|A\|, \qquad A^{**} = A,\ \ and\ \|A\| = \|A^*\|.$$

Theorems:

- A compact -> A* compact.
- A finite rank has A* finite rank.

Lemma: $A\ lin\ bnd, then\ (\ker(A))^{\perp} = \overline{ran}(A^*)$. So $H = \overline{ran}(A^*) \oplus \ker(A)$.
This is especially nice for self-adjoint operators, which will be useful later.

Theorem: Max-Milgram

If we have:

- $H\ Hilbert, V \subseteq H\ \ a\ linear\ subspace\ s.t.\ \ (V, <.,.>_V)\ is\ also\ complete,$
- $\exists C \geq 0\ \ s.t.\ \ \|v\|_H \leq C\|v\|_V,$
- $\exists a: VxV \to \mathbb{K}\ sesquilinear\ \ s.t.$
  - $a\ is\ bnd, |a(u,v)| \leq c\|u\|\|v\|$
  - $a\ is\ coercive, \exists\delta > 0\ \ s.t.\ \ |a(u,v)| \geq \delta\|u\|_V^2,$

Then $\forall f \in H, \exists!\, u \in V\ \ s.t.\ \ a(u,v) = < f, v >_H,\ \forall v \in V.$

Even, $A: H \to V\ defined\ by\ \ Af \coloneqq u$ has norm $\|A\| \leq C/\delta$

Approximate eigenvalues

Def $for\ A \in \mathcal{L}(E),\ \lambda \in \mathbb{C}\ is\ an\ approximate\ eigenvalue\ \tilde{\lambda}\ if\ \exists(f_n) \in E\ s.t.$

$\|f_n\| = 1,$ & $\|\lambda f_n - Af_n\| \to 0.$

Here the sequence of functions can be understood as approximate eigenvectors.

Lemma: $Let\ A\ be\ bnd\ on\ E\ Banach.\ If\ (\sigma I - A)\ is\ invertible,\ then\ \sigma\ cannot\ be\ an\ \tilde{\lambda}.$
Even, $if\ |\sigma| > \|A\|,\ then\ \sigma I - A\ is\ invertible\ (and,\ \sigma\ is\ no\ approx\ eigvalue)$

Theorem:
$Let\ A \in \mathcal{L}(E)\ for\ E\ Banach.\ Then\ if\ \lambda! = 0\ is\ an\ approx\ eigenvalue,$
$then\ it\ is\ an\ eigenvalue.$

Furtermore, $\dim\ker(\lambda I - A) < \infty.$

Self-Adjoint: A*=A. THEN:
- $< Af, f > \in \mathbb{R}$
- $\|A\| = |\|A\|| := \sup\{|< Af, f >|\ \ s.t.\ \ \|f\| = 1\}.$ THEN

- All eigenvalues of A are real.
- All eigenvectors are orthogonal.
- $F\ a\ subspace\ of\ H\ s.t.\ A(F) \subseteq F \to\ A(F^{\perp}) \subseteq F^{\perp}$, i.e. $\forall f \in F:\ Af \in F$
- $\boldsymbol{A = A^*\ compact\ on\ H,\ then\ \exists\lambda \in \mathbb{R}\ s.t.\ \|A\| = |\lambda|.}$

Examples: orthogonal projections, multiplication op on l^{∞}, HS integral operators if $\overline{k(x,y)} = k(y,x)$

Theorem: SPECTRAL THEOREM: $For\ A = A^* \in \mathcal{L}(H)\ compact.$
Then $\exists(e_n)$ with some indexing set J such that :

- $(e_n)\ is\ an\ ONS$
- $\exists(\lambda_n)\ all\ in\ \mathbb{R}/\{0\}$ with $\lambda_n \to 0$ (in case J = N), s.t.:
- $\boldsymbol{\forall x \in H, Ax = \sum \lambda_n < x, e_n > e_n}$. **Even:** $\boldsymbol{Ae_j = \lambda e_j}$

Uniform Boundedness

$\boldsymbol{Def}: a\ collection\ \mathcal{T}\ of\ linear\ op's: E \to F\ \ is\ \boldsymbol{uniformly\ bounded}\ if\ \exists c \geq 0:$
$\|Tf\| \leq c\|f\|,\ \forall f \in E, T \in \mathcal{T}.$

In other words, $\mathcal{T}$ is uniformly bounded if each T is bounded and $\sup\{\|T\|\} < \infty$.
Once can view $\mathcal{T} \subseteq \mathcal{L}(E;F)$ as a bounded subset.

$\boldsymbol{Def}: \mathcal{T}\ is\ pointwise\ bnd\ if\ \|Tf\| \leq \sup_{S \in \mathcal{T}}\{\|S\|\}\,\|f\|.$

Theorem: $E\ Banach, F\ normed \to\ \mathcal{T}\ is\ uniformly\ bounded\ iff\ it\ is\ pointwise\ bnd.$

Theorem 15.6 (Banach–Steinhaus[2]). *Let E, F be Banach spaces, and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$ be a sequence such that*

$$Tf := \lim_{n\to\infty} T_n f$$

exists for every $f \in E$. Then T is a bounded operator, $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, and

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|\,.$$

Important theorems

General analysis – finite dimensional

Projection and orthonormal system

Lemma 1.10. *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space with associated norm $\|\cdot\|$, and let $e_1, \dots, e_n \in E$ be a finite orthonormal system.*

a) *Let $g = \sum_{j=1}^n \lambda_j e_j$ (with $\lambda_1, \dots, \lambda_n \in \mathbb{K}$) be any linear combination of the e_j. Then*

$$\langle g, e_k\rangle = \sum_{j=1}^n \lambda_j \langle e_j, e_k\rangle = \lambda_k \qquad (k = 1, \dots, n)$$

and $$\|g\|^2 = \sum_{j=1}^n |\lambda_j|^2 = \sum_{j=1}^n |\langle g, e_j\rangle|^2.$$

b) *For $f \in E$ let $Pf := \sum_{j=1}^n \langle f, e_j\rangle e_j$. Then*

$$f - Pf \perp \operatorname{span}\{e_1, \dots, e_n\} \quad \text{and} \quad \|Pf\| \le \|f\|.$$

Bessel's inequality

Combining a) and b) of Lemma 1.10 one obtains **Bessel's inequality**[5]

$$\sum_{j=1}^n |\langle f, e_j\rangle|^2 = \|Pf\|^2 \le \|f\|^2 \qquad (f \in E). \tag{1.1}$$

Also extends to inf dim on Hilbert spaces.

~Projection properties

Exercise 1.8. Let $\{e_1, \dots, e_n\}$ be a finite orthonormal system in an inner product space $(E, \langle\cdot,\cdot\rangle)$, let $F := \operatorname{span}\{e_1, \dots, e_n\}$ and let $P : E \to F$ be the orthogonal projection onto F. Show that the following assertions hold:

a) $PPf = Pf$ for all $f \in E$.

b) If $f, g \in E$ are such that $g \in F$ and $f - g \perp F$, then $g = Pf$.

c) Each $f \in E$ has a *unique* representation as a sum $f = u + v$, where $u \in F$ and $v \in F^\perp$. (In fact, $u = Pf$.)

d) If $f \in E$ is such that $f \perp F^\perp$, then $f \in F$. (Put differently: $(F^\perp)^\perp = F$.)

e) Let $Qf := f - Pf$, $f \in E$. Show that $QQf = Qf$ and $\|Qf\| \le \|f\|$ for all $f \in E$.

Cauchy-Schwarz

Theorem 2.1 (Cauchy–Schwarz Inequality[1,2]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space with associated norm $\|f\| := \sqrt{\langle f, f\rangle}$ for $f \in E$. Then*

$$|\langle f, g\rangle| \le \|f\|\,\|g\| \qquad (f, g \in E),$$

with equality if and only if f and g are linearly dependent.

For the proof the following is considered:

$$P : E \longrightarrow \operatorname{span}\{g\}, \qquad Pf := \frac{\langle f, g\rangle}{\|g\|^2} g$$

For questions of the form

$||fg|| \le c||f||$, try to write to a form

$|\langle f, g\rangle|^2 \le ||f||^2||g||^2$, i.e. C-S.

Triangle inequalities

$$||f + g|| \le ||f|| + ||g||,$$

$$|\,||f|| - ||g||\,| \le ||f - g||$$

Lebesque and infinite dimensions

On the Lebesque integral: Dominated convergence

Theorem 7.16 (Dominated Convergence Theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}^1(X)$ such that $f := \lim_{n\to\infty} f_n$ exists pointwise almost everywhere. If there is $0 \le g \in \mathrm{L}^1(X)$ such that $|f_n| \le g$ almost everywhere, for each $n \in \mathbb{N}$, then $f \in \mathrm{L}^1(X)$, $\|f_n - f\|_1 \to 0$ and*

$$\int_X f_n \, d\lambda \to \int_X f \, d\lambda.$$

On the inf dimensional projection operator

Theorem 8.8. *Let F be a closed subspace of a Hilbert space H. Then the orthogonal projection P_F has the following properties:*

a) $P_F f \in F$ *and* $f - P_F f \perp F$ *for all* $f \in H$.

b) $P_F f \in F$ *and* $\|f - P_F f\| = d(f, F)$ *for all* $f \in H$.

c) $P_F : H \to H$ *is a bounded linear mapping satisfying* $(P_F)^2 = P_F$ *and*

$$\|P_F f\| \le \|f\| \qquad (f \in H).$$

In particular, either $F = \{0\}$ or $\|P_F\| = 1$.

d) $\operatorname{ran}(P_F) = F$ *and* $\ker(P_F) = F^\perp$.

e) $\mathrm{I} - P_F = P_{F^\perp}$, *the orthogonal projection onto* $F^\perp$.

Riesz-Fréchet

Theorem 8.12 (Riesz–Fréchet[1]). *Let H be a Hilbert space and let $\varphi : H \to \mathbb{K}$ be a bounded linear functional on H. Then there exists a unique $g \in H$ such that*

$$\varphi(f) = \langle f, g\rangle \qquad \text{for all } f \in H.$$

Decomposition of L^2

Lemma 10.5. *The space $\mathrm{L}^2(a,b)$ decomposes orthogonally into*

$$\mathrm{L}^2(a,b) = \mathbb{C}\mathbf{1} \oplus \overline{\{\psi' \mid \psi \in \mathrm{C}_0^1[a,b]\}},$$

with $\|\cdot\|_2$-closure on the right-hand side.

Note that C1 is the space of constant functions.
Gives as **corollary the fundamental thm of calc:**

Corollary 10.7. *One has $\mathrm{H}^1(a,b) \subseteq \mathrm{C}[a,b]$. More precisely, $f \in \mathrm{H}^1(a,b)$ if and only if f has a representation*

$$f = Jg + c\mathbf{1}$$

with $g \in \mathrm{L}^2(a,b)$ and $c \in \mathbb{K}$. Such a representation is unique, namely

$$g = f' \quad \text{and} \quad c = \frac{\langle f - Jf', \mathbf{1}\rangle}{b-a}.$$

Moreover, the **fundamental theorem of calculus** *holds, i.e.,*

$$\int_c^d f'(s)\, ds = f(d) - f(c) \quad \text{for every interval } [c,d] \subseteq [a,b].$$

Operator theory:

Theorem 11.2. *Let $1 \le p < \infty$. If $f \in L^p(X)$ and $g \in L^p(Y)$, then $f \otimes g \in L^p(X \times Y)$ with $\|f \otimes g\|_{L^p(X\times Y)} = \|f\|_{L^p(X)} \|g\|_{L^p(Y)}$. Moreover, the space*

$$\operatorname{span}\{f \otimes g \mid f \in L^p(X),\, g \in L^p(Y)\}$$

is dense in $L^p(X \times Y)$.

Fubini

The integral of an integrable function $f \in L^1(X \times Y)$ with respect to two-dimensional Lebesgue measure is computed via iterated integration in either order:

$$\int_{X\times Y} f(\cdot,\cdot)\, d\lambda^2 = \int_X \int_Y f(x,y)\, dy\, dx.$$

This is called **Fubini's theorem**[1] and it includes the statement that if one integrates out just one variable, the function

$$x \longmapsto \int_Y f(x,y)\, dy$$

is again measurable.

Ez integration

Lemma 11.3. *Let $f \in L^1(a,b)$ and $n \in \mathbb{N}$. Then*

$$(J^n f)(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)\, ds \quad \text{for all } t \in [a,b].$$

In particular, J^n is again an integral operator, with kernel function

$$k_n(t,s) = \frac{1}{(n-1)!}\mathbf{1}_{[a,t]}(s)(t-s)^{n-1} \qquad (s,t \in [a,b]).$$

Proof. This is proved by induction and Fubini's theorem. □

Example 11.9 (Integration Operator). The n-th power of the integration operator J on $E = C[a,b]$ is induced by the integral kernel

$$k_n(t,s) = \mathbf{1}_{\{s \le t\}}(t,s)\frac{(t-s)^{n-1}}{(n-1)!}.$$

From this it follows that $\|J^n\|_{\mathcal{L}(E)} = 1/n! \ne 1^n = \|J\|^n$. (See Exercise 11.7.)

The above uses HS-operators from the operator summary.

Lax-Milgram:

Let $a : V \times V \to \mathbb{K}$ be a sesquilinear mapping with the following properties:

1) a is **bounded**, i.e., there is $c > 0$ such that

(12.6) $$|a(u,v)| \le c\|u\|_V \|v\|_V \qquad (u,v \in V).$$

2) a is **coercive**, i.e., there is $\delta > 0$ such that

$$|a(u,u)| \ge \delta \|u\|^2 \qquad (u \in V).$$

(The number δ is called the **coercivity constant**.)

Then we have the following theorem.

Theorem 12.13 (Lax–Milgram[4,5]). *In the situation described above, for each $f \in H$ there is a unique $u \in V$ such that*

$$a(u,v) = \langle f, v\rangle_H \quad \text{for all } v \in V.$$

Moreover, the operator $A : H \to V$ defined by $Af := u$ has norm $\|A\| \le C/\delta$.

Spectral Theorem

Theorem 13.11 (Spectral Theorem). *Let A be a compact self-adjoint operator on a Hilbert space H. Then A is of the form*

(13.1) $$Af = \sum_j \lambda_j \langle f, e_j\rangle e_j \qquad (f \in H)$$

for some (finite or countably infinite) orthonormal system $(e_j)_j$ and real numbers $\lambda_j \ne 0$ satisfying $\lim_{j\to\infty} \lambda_j = 0$. Moreover, $Ae_j = \lambda e_j$ for each j.

More precisely, the orthonormal system is either $(e_j)_{j=1}^N$ for some $N \in \mathbb{N}$ or $(e_j)_{j\in\mathbb{N}}$. Of course, the condition $\lim_{j\to\infty}\lambda_j = 0$ is only meaningful in the second case.

In fact for $A = A^*$ and the spectral thm we get that A is characterized somewhat by a projection:

Let us denote by J the index set for the orthonormal system in the spectral theorem. So $J = \{1,\dots,N\}$ or $J = \mathbb{N}$. Moreover, let

$$P_0 : H \longrightarrow \ker(A)$$

be the orthogonal projection onto the kernel of A and $P_r := I - P_0$ its complementary projection. Then we can write

$$Af = 0 \cdot P_0 f + \sum_{j\in J} \lambda_j \langle f, e_j\rangle e_j$$

for all $f \in H$. This formula is called the **spectral decomposition** of A.

Corollary 13.12. *Let A be as in the spectral theorem (Theorem 13.11). Then the following assertions hold.*

a) $\overline{\operatorname{ran}}(A) = \overline{\operatorname{span}}\{e_j \mid j \in J\}$ *and* $\ker(A) = \{e_j \mid j \in J\}^\perp$.

b) $P_r f = \sum_{j\in J}\langle f, e_j\rangle e_j$ *for all $f \in H$.*

c) *Every nonzero eigenvalue of A occurs in the sequence $(\lambda_j)_{j\in J}$, and its geometric multiplicity is*

$$\dim\ker(\lambda I - A) = \operatorname{card}\{j \in J \mid \lambda = \lambda_j\} < \infty.$$

Baire:

Lemma:

$$Let\ (\Omega, d) be\ complete, \qquad B_i = B[x_i, r_i],$$
$$B_1 \supseteq B_2 \supseteq B_3 \ldots$$

Be a nested sequence of closed balls. $If\ r_n \to 0,$ $then\ x_n \to x\ exists,\ and\ \bigcap_n B_n = \{x\}.$

Theorem 15.1 (Baire). *Let (Ω, d) be a nonempty complete metric space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of Ω such that*

$$\Omega = \bigcup_{n\in\mathbb{N}} A_n.$$

Then there is $n \in \mathbb{N}$ and $x \in \Omega, r > 0$ with $B(x,r) \subseteq A_n$.

Alternatively:

1. $If\ \exists (O_n) \subseteq \Omega\ open\ s.t.\ \overline{O_n} = \Omega\ \forall n \in \mathbb{N}$ $\to \bigcap O_n \ne 0.$
2. $If\ \exists O_n \subseteq \Omega\ open\ s.t.\ \overline{O_n} = \Omega\ \forall n \in \mathbb{N}$ $\to \overline{\bigcap O_n} = \Omega.$

Banach-Steinhaus

Theorem 15.6 (Banach–Steinhaus[2]). *Let E, F be Banach spaces, and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$ be a sequence such that*

$$Tf := \lim_{n\to\infty} T_n f$$

exists for every $f \in E$. Then T is a bounded operator, $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, and

$$\|T\| \le \liminf_{n\to\infty}\|T_n\|.$$

OMP:

Theorem 15.8 (Open Mapping Theorem). *Let E, F be Banach spaces and let $T : E \to F$ be a bounded linear mapping which is surjective. Then there is $a > 0$ such that for each $g \in F$ there is $f \in E$ with $\|f\| \le a\|g\|$ and $Tf = g$.*

Alternatively, T maps open subsets of E onto open subsets of F.

$If\ T\ is\ invertible\ then\ the\ inverse\ is\ bounded\ and\ the\ statement\ also\ holds\ for\ T^{-1}.$

Approximate surjectivity:

Theorem 15.11. *Let E, F be Banach spaces, and let $T \in \mathcal{L}(E; F)$. Suppose that there exist $0 \le q < 1$ and $a \ge 0$ such that for every $g \in F$ with $\|g\| \le 1$ there is $f \in E$ such that*

$$\|f\| \le a \quad \text{and} \quad \|Tf - g\| \le q.$$

Then for each $g \in F$ there is $f \in E$ such that $Tf = g$ and $\|f\| \le \frac{a}{1-q} \|g\|$.

So approx. surjectivity implies surjectivity with an estimate of the pre-image.

Closed graph theorem:

Def $T: E \to F$ has a closed graph if

$$\left.\begin{array}{r} f_n \to f \\ Tf_n \to g \end{array}\right\} \to Tf = g \ \forall (f_n), f \in E, g \in F.$$

In other words, if $graph(T) := \{(f, Tf) | \ f \in E\}$ is closed in the normed VS $E \ x \ F$.

THM: $E, F \ Banach, then \ T \ is \ bnd \ iff$
$graph(T) \ is \ closed$

Tietze: (not very enlightening)

Tietze's Theorem. Let (Ω, d) be a metric space. Any subset $A \subseteq \Omega$ is a metric space with respect to the induced metric, and if $f \in C_b(\Omega)$ is a bounded continuous function, one can consider its restriction

$$Tf := f|_A \in C_b(A)$$

to the set A. The operator $T : C_b(\Omega) \to C_b(A)$ is linear with $\|T\| \le 1$. Tietze's theorem states that if A is closed, then T is surjective.

Theorem 15.15 (Tietze[3]). *Let (Ω, d) a metric space, $A \subseteq \Omega$ a closed subset and $g \in C_b(A; \mathbb{R})$. Then there is $h \in C_b(\Omega; \mathbb{R})$ such that $h|_A = g$ and $\|h\|_\infty = \|g\|_\infty$.*

Duality theorems:

Below I conclude with only stuff about duality CH16 since it has to be somewhere, but not in a separate file. Dual def is in the spaces of note.

Does E' always exist (nonzero)?

Often yes: E fin dim, E' same dim
E inner product space, E'=E.

Idea: the dual may be rich enough to distinguish points in E based on evaluation with point in E':
$\forall x \neq y \in E, \exists \varphi \in E' s.t. \ \varphi(x) \neq \varphi(y).$

Theorem $H \ Hilbert \to H' isometrically \ isomorphihc \ to \ H$. Think of row/col vectors.

Your best mates Riesz-Fréchet say then, as proof:
$h, g \in H, any \ \varphi \in H' \sim <., m> for \ some \ m \in H.$
$\varphi(h) = <h, m> \neq \varphi(g) iff \ <h - g, m> \neq 0$

General case: Hahn-Banach

For $(E, \|.\|), E_0 \subseteq E \ \& \ \varphi_0 \in E_0'$. Then
$\exists \varphi \in E' \ s.t. \varphi(f) = \varphi_0(f) \ \forall \ f \in E_0,$

$\& \ \|\varphi\|_{E'} = \|\varphi\|_{E_0'}.$

Cor: every Hilbert space H with countable basis is separable.

Corollaries:
- $\forall f \in E, \exists \ \varphi \in E' s.t. \ \|\varphi\| = 1 \ and \ |\varphi(f)| = \|f\|.$
- $\forall f \in E, \|f\| = \sup_{\|\varphi\|=1} |\varphi(f)|$
- $\forall f \subseteq E, have \ \overline{span}(A) = E \ \ iff$
  $\forall \varphi \in E' \ we \ have \ \ \varphi|_A = 0 \to \varphi = 0$

If H is a Hilbert space then:

- $\forall f \in H \ \exists \|g\| = 1 \in H : \|f\| = |<f, g>|$
- $\|f\| = \sup_{\|g\|=1} |<f, g>|$ obviously,
- $\overline{span}(A) = H \ \ iff \ \ \forall g \in H \ have$
  $<f, g> = 0 \ \forall f \in A \to g = 0.$

This last one can be restated as
$A^\perp = \{0\} \ iff \ \ g \in H \ \ with \ \ g \in A^\perp \to g = 0.$ Trivial.

I cannot be bothered with the pf and the further corollaries.

Sobolev and Poisson

*Note: this document and the one on operators are heavily linked. I want to keep the document on operators and spaces as general as possible. Therefore, this document implicitly draws results/facts from the others.

$$u'' = -f, \;\; u(a) = u(b) = 0$$

To solve this, we need to move to Lebesque spaces and use the **weak derivative:**

Def: weak derivative:

$g \in L^2(a,b)$ is said to be a weak derivative of $f \in L^2(a,b)$ if they satisfy

$\int_a^b g\psi ds = -\int_a^b f\psi' ds$ holds for every test function $\psi \in C_0^1[a,b]$.
This can be rewritten as $< g, \psi > = -< f, \psi' >$

We call the space of all weakly differentiable functions $H^1(a,b)$, the first order Sobolev space.

Variational method for Poisson

We could say that $u \in H^2(a,b)$ since we have a second derivative.

Now rewrite Poisson to $< \psi', u' >_{L^2} = < \psi, f >_{L^2}, \qquad \psi \in C_0^1[a,b]$.

We now constrain u to be in the space H_0^1, which is defined as you would expect, with norm

$$||u||_{H_0^1} := ||u'||_{L^2}$$

Then we rewrite the RHS by using $\varphi: H_0^1(a,b) \to \mathbb{C}, \;\; \varphi(v) := < v, f >_{L^2}$

Then Riesz-Fréchet yields a unique $u \in H_0^1(a,b)$ such that

$$< v', u' >_2 =: < v, u >_{H_0^1} = \varphi(v) = < v, f >_2$$

For all $v \in H_0^1(a,b)$. In short, there is a u s.t. $< v', u' >_2 = < v, f >_2$, which holds for all $v \in H_0^1 \supset C_0^1[a,b]$, as required for our problem.

Dirichlet-Laplacian & Hilbert-Schmidt

Def: Dirichlet-Laplacian: $\Delta_D: H_0^2(a,b) \to L^2(a,b), \qquad \Delta_D u := u''$

The importance of writing this as an operator is that there is an inverse operator $\Delta_D^{-1}: L^2 \to H^2$ that turns out to be a HS (kernel) integral operator, which turns out to be bounded, which means that the Poisson problem is well-posed. This is because $-\Delta_D^{-1}$ maps the problem to its unique solution.

Perturbations

$u'' - Tu = -f, \qquad u \in H_0^2(a,b)$. It turns out T 'small enough' is still well-posed.

We use the property that $(\Delta_D \ bijective\ and\ bounden) + (Poisson\ well-posed)$

$\rightarrow I - T\Delta_D^{-1}: L^2 \rightarrow L^2\ is\ invertible.$

Note that we can rewrite this problem using the inverse Dirichlet as $-f = (I - T\Delta_D^{-1})\Delta_D u$.

Lemma

Now we can just look at conditions s.t. $(I - A)u = f$ has a unique solution, for $A \in \mathcal{L}(E)$ a perturbation. Without too much work I note that $if\ \ f \in E\ is\ s.t.\ \ u := \sum A^n f\ \ converges\ in\ E,$ $then\ \ u - Au = f.$

Theorem from the above, $\sum \|A^n\| < \infty, \rightarrow\ \ (I - A)\ is\ invertible\ with$

$(I - A)^{-1} = \sum A^n$, the **Neumann series**.

Returning to our problem, **the perturbation is still well-posed if** $\|T\Delta_D^{-1}\| < 1$.

Then in the book there is Volterra which I skip here.

Using compact-self adjoint & Spectral theorem

We can consider the general **eigenvalue equation** $Au - \lambda u = f$

Where $f \in H\ \ Hilbert,\ \lambda \in \mathbb{K},\ A$ is compact self-adjoint. This is solvable under the following theorem (with (e_j) from the spectral theorem):

Theorem 13.13 (Fredholm Alternative[1]). *In the situation above, precisely one of the following cases holds:*

1) *If $\lambda \neq 0$ is different from every λ_j, then $(\lambda \mathrm{I} - A)$ is invertible and*

$$u := (A - \lambda \mathrm{I})^{-1} f = -\frac{1}{\lambda} P_0 f + \sum\nolimits_{j \in J} \frac{1}{\lambda_j - \lambda} \langle f, e_j \rangle e_j$$

is the unique solution to (13.2).

2) *If $\lambda \neq 0$ is an eigenvalue of A, then (13.2) has a solution if and only if $f \perp \ker(\lambda \mathrm{I} - A)$. In this case a particular solution is*

$$u := -\frac{1}{\lambda} P_0 f + \sum\nolimits_{j \in J_\lambda} \frac{1}{\lambda_j - \lambda} \langle f, e_j \rangle e_j,$$

where $J_\lambda := \{j \in J \mid \lambda_j \neq \lambda\}$.

3) *If $\lambda = 0$, then (13.2) is solvable if and only if $f \in \operatorname{ran}(A)$; in this case one particular solution is*

$$u := \sum\nolimits_{j \in J} \frac{1}{\lambda_j} \langle f, e_j \rangle e_j,$$

this series being indeed convergent.

Let us consider then $\Delta_D u = -f$, with its solution $-\Delta_D^{-1} f = Af = \int g(.,s) f(s) ds,$ $g(s)$ the Green function. Note the following:

- $A\ is\ a\ HS - operator\ and\ hence\ compact$
- $k\ is\ symmetric\ and\ real - valued, hence\ A\ is\ self\ adjoint$
- $\ker(A) = \{0\}$ by construction.

Hence: we can apply the spectral theorem if we can find the eigenvalues and eigenvectors of A.

So first, to determine the eigenvalues/eigenvectors:

Lemma 14.1. *Let $\lambda \neq 0$, $\mu = -1/\lambda$. Then*

$$f \in L^2(a,b) \quad and \quad Af = \lambda f \quad \Longleftrightarrow \quad f \in \mathrm{dom}(\Delta_D) \quad and \quad \Delta_D f = \mu f.$$

Moreover, in this case either $f = 0$ or $\lambda > 0$.

Where $dom(\Delta_D) = H_0^2(a,b)$.

By ez DE-theory we have $Au = \lambda u, \lambda > 0 \; iff$

$u = \alpha \cos\left(\frac{t}{\sqrt{\lambda}}\right) + \beta \sin\left(t/\sqrt{\lambda}\right)$. Now this solution can be further sharpened by the boundary conditions. In particular, letting $a = 0, b = 1, u(0) = 0 \to \alpha = 0$.
Then $u(1) = 0, \beta \neq 0 \to \sin\left(\frac{t}{\sqrt{\lambda}}\right) = 0 \quad \to \quad \lambda_n = \frac{1}{n^2\lambda^2}, e_n = \frac{1}{\sqrt{2}} \sin(n\pi t)$ (normalized).

Furthermore, A is injective to L^2, so the system (e_n) is an orthonormal basis for $L^2(0,1)$, and:

$$(Af)(t) = \int_0^1 g(t,s) f(s)\, \mathrm{d}s = \sum_{n=1}^{\infty} \left(\frac{1}{2n^2\pi^2} \int_0^1 f(s) \sin(n\pi s)\, \mathrm{d}s \right) \sin(n\pi t)$$

Which converges by the theory in L^2 but not necessarily pointwise. However it can be shown that it does in fact converge uniformly in t=[0,1]. Furthermore:

$$g(t,s) = \sum_{n=1}^{\infty} \frac{\sin(n\pi \cdot t)\, \sin(n\pi \cdot s)}{2n^2\pi^2}$$

As an absolutely convergent series in $C([0,1]x[0,1])$.

Schrödinger operator & Strum Liouville equation

Is just a perturbation of the Dirichlet-Laplacian with a multiplication operator:

$Lu = -u'' + qu \;\; for\ some\ q \in C[0,1]$ a positive continuous function, called the potential. Once again the domain is $dom(L) = H_0^2(0,1)$. We can consider the eigenvalues of L.

$Lu = \lambda u,\ then\ \ u \in C^2[0,1]\ and\ either\ \ u = 0\ \ or\ \lambda < 0$. In particular, $L\ is\ injective$ (1-1).

Sturm-Liouville:

$Lu = f$, is well-posed for $f \in L^2(0,1)$, i.e., $L: H_0^1 \to L^2$ is bijective with bounded inverse. To this end, we define the new inner product $a(u,v) :=< u', v' >_2 + < qu, v >_2$. The induced norm is equivalent to the usual norm on $H_0^1, \quad and\ (H_0^1, \|.\|_a)$ is a Hilbert space. Then the mapping

$v \mapsto < v, f >_2$ is bounded, and by Riesz-Fréchet $\exists!\, u \in H_0^1 \;\; s.t. \;\; a(u,v) = < f, v > \forall v \in H_0^1$.

For $v \in C_0^1, then,\ \ u \in H_0^2,\ \ and\ \ Lu = f,\ \ and\ L\ is\ bijective$. It can also be shown that L^{-1} is bnd. In the book they show L^{-1} can be found (as a HS-integral operator) but that is cumbersome and skipped.

Fourier analysis

From Chapter 1:

The number

$$\widehat{f}(n) := \langle f, e_n \rangle = \int_0^1 f(t)\overline{e_n(t)}\,\mathrm{d}t = \int_0^1 f(t)\mathrm{e}^{-2\pi i n t}\,\mathrm{d}t$$

is called the n-th **Fourier coefficient** of f. Note that n ranges over the whole set of integers $\mathbb{Z}$. Bessel's inequality in this context reads

$$\sum_{n=-N}^{N} |\widehat{f}(n)|^2 \le \|f\|_2^2 = \int_0^1 |f(t)|^2\,\mathrm{d}t. \tag{1.3}$$

With $e_n(t) = e^{2n\pi i t}$

This can be extended after Lebesque and Hilbert to inf dim:

For a function $f \in \mathrm{L}^1(\mathbb{R})$ its **Fourier transform** $\mathcal{F}f$ is defined by

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}} f(s)\,\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s \qquad (t \in \mathbb{R}). \tag{9.3}$$

The integral is well-defined since

$$\int_{\mathbb{R}} \left| f(s)\,\mathrm{e}^{-\mathrm{i}st} \right| \,\mathrm{d}s = \int_{\mathbb{R}} |f(s)|\,\mathrm{d}s = \|f\|_1 < \infty.$$

Moreover, by the triangle inequality for integrals it follows that $|(\mathcal{F}f)(t)| \le \|f\|_1$ and taking the supremum over $t \in \mathbb{R}$ we arrive at

$$\|\mathcal{F}f\|_\infty \le \|f\|_1 \qquad (f \in \mathrm{L}^1(\mathbb{R})). \tag{9.4}$$

This shows that the Fourier transform is a bounded linear operator

$$\mathcal{F} : (\mathrm{L}^1(\mathbb{R}), \|\cdot\|_1) \longrightarrow (\mathcal{B}(\mathbb{R}), \|\cdot\|_\infty).$$

Applying the dominated convergence theorem one can show that $\mathcal{F}f$ is a continuous function for every $f \in \mathrm{L}^1(\mathbb{R})$; see Exercise 7.21. Regarding the asympotic behaviour of $\mathcal{F}f(t)$ for large values of $|t|$ we have the following analogue of Theorem 9.19.

Theorem 9.20 (Riemann–Lebesgue[5]). *If $f \in \mathrm{L}^1(\mathbb{R})$, then $\mathcal{F}f \in \mathrm{C}(\mathbb{R})$ and*

$$\lim_{|t|\to\infty} (\mathcal{F}f)(t) = 0.$$

Theorem 15.7 (Du Bois-Reymond). *There exists a function $f \in \mathrm{C}_{\mathrm{per}}[0,1]$ such that its partial Fourier series at $t = 0$,*

$$S_n f(0) = \sum_{k=-n}^{n} \widehat{f}(k) e^{2\pi i k t}\Big|_{t=0} = \sum_{k=-n}^{n} \widehat{f}(k) \qquad (n \in \mathbb{N})$$

does not converge to $f(0)$.

Need the Dirichlet kernel:

Define the **Dirichlet kernel**

$$D_n(s) := \frac{\sin(2n+1)\pi s}{\sin \pi s},$$

so that $T_n f = \int_0^1 D_n(s) f(s)\,\mathrm{d}s$ for $f \in E$. We claim that

$$\|T_n\| = \int_0^1 |D_n(s)|\,\mathrm{d}s.$$

Proof. We consider the linear functionals

$$T_n : \mathrm{C}_{\mathrm{per}}[0,1] \longrightarrow \mathbb{C}, \qquad T_n f := (S_n f)(0)$$

for $n \in \mathbb{N}$. Then

$$T_n f = \sum_{k=-n}^{n} \int_0^1 \mathrm{e}^{2\pi i k s} f(s)\,\mathrm{d}s = \int_0^1 \frac{\sin(2n+1)\pi s}{\sin \pi s} f(s)\,\mathrm{d}s$$

Then some stuff and some more stuff with which I can't be bothered and then $\|T_n\|$ is the harmonic series which diverges.