Afa – Applied Functional Analysis

Know-by-heart

Solutions

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Lemma 1.11 (Gram[6]–Schmidt[7]). *Let $N \in \mathbb{N} \cup \{\infty\}$ and let $(f_n)_{1 \le n < N}$ be a linearly independent set of vectors in an inner product space E. Then there is an orthonormal system $(e_n)_{1 \le n < N}$ in E such that*

$$\operatorname{span}\{e_j \mid 0 \le j < n\} = \operatorname{span}\{f_j \mid 0 \le j < n\} \qquad \textit{for all } n \le N.$$

Proof. The construction is recursive. By the linear independence, f_1 cannot be the zero vector, so $e_1 := (1/\|f_1\|) f_1$ has norm one. Let $g_2 := f_2 - \langle f_2, e_1 \rangle e_1$. Then $g_2 \perp e_1$. Since f_1, f_2 are linear independent, $g_2 \neq 0$ and so $e_2 := (1/\|g_2\|) g_2$ is the next unit vector.

Suppose that we have already constructed pairwise orthogonal unit vectors $e_1, \ldots, e_{n-1}$ such that $\operatorname{span}\{e_1, \ldots, e_{n-1}\} = \operatorname{span}\{f_1, \ldots, f_{n-1}\}$. If $n = N$, we are done. Otherwise let

$$g_n := f_n - \sum_{j=1}^{n-1} \langle f_n, e_j \rangle e_j.$$

Then $g_n \perp e_j$ for all $1 \le j < n$ (Lemma 1.10). Moreover, by the linear independence of the f_j and the construction of the e_j so far, $g_n \neq 0$. Hence $e_n := (1/\|g_n\|) g_n$ is the next unit vector in the orthonormal system. □

Theorem 2.1 (Cauchy–Schwarz Inequality[1,2]). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space with associated norm $\|f\| := \sqrt{\langle f, f \rangle}$ for $f \in E$. Then*

$$|\langle f, g \rangle| \le \|f\| \, \|g\| \qquad (f, g \in E),$$

with equality if and only if f and g are linearly dependent.

Proof. If $g = 0$, the inequality reduces to the trivial identity $0 = 0$. So suppose that $g \neq 0$ and consider the orthogonal projection

$$P : E \longrightarrow \operatorname{span}\{g\}, \qquad Pf := \frac{\langle f, g \rangle}{\|g\|^2} g$$

of E onto the one-dimensional subspace of E spanned by g; cf. page 8. By Lemma 1.10 we have $f - Pf \perp g$ and hence, by Pythagoras' Lemma 1.9,

$$\|f\|^2 = \|Pf\|^2 + \|f - Pf\|^2 = \frac{|\langle f, g \rangle|^2}{\|g\|^4} \|g\|^2 + \|f - Pf\|^2 \ge \frac{|\langle f, g \rangle|^2}{\|g\|^2}$$

with equality if and only if $f = Pf$, i.e., $f \in \operatorname{span}\{g\}$. □

Example of bounded linear mapping?

➔ Any linear mapping from $K^d \to F$, F arbitrary normed space.

Pf: Use Cauchy schwartz

Example 3.12 (Scale of ℓ^p-Spaces). Let $f : \mathbb{N} \to \mathbb{K}$ be any scalar sequence. We claim that

$$\|f\|_\infty \leq \|f\|_2 \leq \|f\|_1 \qquad \text{in } [0, \infty]. \tag{3.5}$$

For the first inequality we note that we have $|f(j)| \leq \|f\|_2$ for every $j \in \mathbb{N}$, and hence we can take the supremum over j to obtain $\|f\|_\infty \leq \|f\|_2$. For the second inequality we estimate

$$\|f\|_2^2 = \sum_{j=1}^{\infty} |f(j)|^2 = \sum_{j=1}^{\infty} |f(j)| \cdot |f(j)| \leq \sum_{j=1}^{\infty} |f(j)| \cdot \|f\|_1 = \|f\|_1^2$$

Example 3.14 (p-Norms on $\mathrm{C}[a, b]$). *For each interval* $[a, b] \subseteq \mathbb{R}$ *we have*

$$\|f\|_1 \leq \sqrt{b-a}\, \|f\|_2 \quad \text{and} \quad \|f\|_2 \leq \sqrt{b-a}\, \|f\|_\infty \tag{3.6}$$

for all $f \in \mathrm{C}[a, b]$.

Proof. The first inequality follows from Cauchy–Schwarz and

$$\|f\|_1 = \int_a^b |f| = \langle |f|, \mathbf{1} \rangle \leq \|f\|_2 \, \|\mathbf{1}\|_2 = \sqrt{b-a}\, \|f\|_2 \, ,$$

where we have written $\mathbf{1}$ for the function which is constantly equal to 1. The second inequality follows from

$$\|f\|_2^2 = \int_a^b |f|^2 \leq \int_a^b \|f\|_\infty^2 \mathbf{1} = (b-a)\, \|f\|_\infty^2 \, . \qquad \square$$

Exercise 3.4. Show that the inclusions

$$\ell^1 \subseteq \ell^2 \subseteq \ell^\infty,$$

are all strict. Give an example of sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ in ℓ^1 with

$$\|f_n\|_\infty \to 0, \ \|f_n\|_2 \to \infty \quad \text{and} \quad \|g_n\|_2 \to 0, \ \|g_n\|_1 \to \infty.$$

Lemma 4.9 (Second triangle inequality). *Let (Ω, d) be a metric space. Then*

$$|d(x,z) - d(y,w)| \le d(x,y) + d(z,w) \qquad \textit{for all } x,y,z,w \in \Omega.$$

Proof. The triangle inequality $d(x,z) \le d(x,y) + d(y,w) + d(w,z)$ yields

$$d(x,z) - d(y,w) \le d(x,y) + d(w,z).$$

Interchanging the roles of x, y and z, w leads to

$$-(d(x,z) - d(y,w)) = d(y,w) - d(x,z) \le d(y,x) + d(z,w).$$

Combining both inequalities concludes the proof. □

Theorem 4.16. *A linear mapping $T : E \to F$ between two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ is continuous if and only if it is bounded.*

Proof. Suppose that T is bounded. Then, if $f_n \to f$ in E is an arbitrary convergent sequence in E,

$$\|Tf_n - Tf\|_F = \|T(f_n - f)\|_F \le \|T\|_{F\leftarrow E} \|f_n - f\|_E \to 0$$

as $n \to \infty$. So $Tf_n \to Tf$, hence T is continuous.

For the converse, suppose that T is *not* bounded. Then $\|T\|$ as defined in (2.2) is not finite. Hence there is a sequence of vectors $(g_n)_{n\in\mathbb{N}}$ in E such that

$$\|g_n\| \le 1 \quad \text{and} \quad \|Tg_n\| \ge n \qquad (n \in \mathbb{N}).$$

Define $f_n := (1/n)g_n$. Then $\|f_n\| = (1/n)\|g_n\| \le 1/n \to 0$, but

$$\|Tf_n\| = \left\|T((1/n)g_n)\right\| = (1/n)\|Tg_n\| \ge 1$$

for all $n \in \mathbb{N}$. Hence $Tf_n \not\to 0$ and therefore T is not continuous. □

Lemma 4.20. *Let $A \subseteq \Omega$ be subset of a metric space (Ω, d). If A is compact, then A is closed in Ω; and if Ω is compact and A is closed in Ω, then A is compact.*

Proof. For the first assertion, suppose that A is compact and let $(x_n)_{n\in\mathbb{N}}$ be a sequence in A with $x_n \to x \in \Omega$. By compactness, there is a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ converging to some element $y \in A$. But also $x_{n_k} \to x$, and as limits are unique, $x = y \in A$. The second statement is left as an (easy) exercise. □

Theorem 4.21 (Bolzano[1]–Weierstrass). *With respect to the Euclidean metric on $\mathbb{K}^d$ a subset $A \subseteq \mathbb{K}^d$ is (sequentially) compact if and only if it is closed and bounded.*

Theorem 4.29. *Let E be a finite dimensional linear space. Then all norms on E are equivalent.*

Proof. By choosing an algebraic basis $e_1, \dots, e_d$ in E we may suppose that $E = \mathbb{K}^d$ and $\{e_1, \dots, e_d\}$ is the canonical basis.

Let $\|\cdot\|$ be any norm on $\mathbb{K}^d$. We shall prove that $\|\cdot\|$ and $\|\cdot\|_2$ are equivalent, the latter being the Euclidean norm. Define

$$m_1 := \left(\sum_{j=1}^d \|e_j\|^2\right)^{1/2}.$$

Then $\|x\| \leq m_1 \|x\|_2$ for all $x \in E = \mathbb{K}^d$; see Example 2.19. By the second triangle inequality

$$\Big|\|x\| - \|y\|\Big| \leq \|x - y\| \leq m_1 \|x - y\|_2$$

we obtain that the norm mapping

$$(\mathbb{K}^d, \|\cdot\|_2) \longrightarrow \mathbb{R}_+, \qquad x \longmapsto \|x\|$$

is continuous. By Bolzano–Weierstrass (Theorem 4.21), the Euclidean unit sphere

$$\mathbb{S}^{d-1} = \{x \in \mathbb{K}^d \mid \|x\|_2 = 1\}$$

is compact. Hence by Corollary 4.24 there is $x' \in \mathbb{S}^{d-1}$ such that

$$\|x'\| = \inf\{\|y\| \mid y \in \mathbb{S}^{d-1}\}.$$

Now, because $\|x'\|_2 = 1$ we must have $x' \neq 0$ and since $\|\cdot\|$ is a norm, $\|x'\| > 0$. By Theorem 4.26, implication (v)⇒(i), we conclude that there is $m_2 \geq 0$ such that

$$\|x\|_2 \leq m_2 \|x\| \quad \text{for all } x \in \mathbb{K}^d.$$

□

Theorem 4.37. *A normed space E is separable if and only if there is a countable set $M \subseteq E$ such that* span(M) *is dense in E.*

Only know by heart.

Corollary 4.34. *In each infinite-dimensional normed space E there is a sequence of unit vectors $(f_n)_{n\in\mathbb{N}}$ such that $\|f_n - f_m\| \geq 1$ for all $n, m \in \mathbb{N}$ with $n \neq m$.*

Only know by heart.

Example 5.10. *Every finite-dimensional normed space is a Banach space.*

Proof. All norms on a finite-dimensional space are equivalent. It is easy to see (Exercise 5.2) that equivalent norms have the same Cauchy sequences. As we know completeness for the Euclidean norm, we are done. □

Example 5.11. *Let Ω be a nonempty set. Then $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$ is a Banach space.*

Note here $\mathcal{B}$ means the set of all bounded functions, not a ball.

Only know by heart

Example 5.13. *The space $\mathrm{C}[a, b]$ is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.*

Only know by heart

Definition 7.1. The **Lebesgue outer measure** of a set $A \subseteq \mathbb{R}$ is

$$\lambda^*(A) := \inf \sum_{n=1}^{\infty} |Q_n|$$

where the infimum is taken over all sequences of intervals $(Q_n)_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} Q_n$. (Such a sequence is called a **cover** of A.)

Know by heart

Theorem 7.16 (Dominated Convergence Theorem). *Let* $(f_n)_{n\in\mathbb{N}}$ *be a sequence in* $\mathrm{L}^1(X)$ *such that* $f := \lim_{n\to\infty} f_n$ *exists pointwise almost everywhere. If there is* $0 \leq g \in \mathrm{L}^1(X)$ *such that* $|f_n| \leq g$ *almost everywhere, for each* $n \in \mathbb{N}$*, then* $f \in \mathrm{L}^1(X)$*,* $\|f_n - f\|_1 \to 0$ *and*

$$\int_X f_n \,\mathrm{d}\lambda \to \int_X f \,\mathrm{d}\lambda.$$

Proof. Note that the *function* f here is defined only almost everywhere. But as such it determines a unique equivalence class modulo equality almost everywhere. It is actually easy to see that $f \in \mathrm{L}^1(X)$: since $f_n \to f$ almost everywhere and $|f_n| \leq g$ almost everywhere, for every $n \in \mathbb{N}$, by "throwing away" countably many null sets we see that $|f| \leq g$ almost everywhere, and hence

$$\int_X |f| \,\mathrm{d}\lambda \leq \int_X g \,\mathrm{d}\lambda < \infty$$

since $g \in \mathrm{L}^1(X)$. So, indeed, $f \in \mathrm{L}^1(X)$.

Second, if we know already that $\|f_n - f\|_1 \to 0$, then the convergence of the integrals is clear from

$$\left| \int_X f_n \,\mathrm{d}\lambda - \int_X f \,\mathrm{d}\lambda \right| = \left| \int_X f_n - f \,\mathrm{d}\lambda \right| \leq \|f_n - f\|_1 \to 0$$

(Lemma 7.8). In other words, the integral is a bounded linear mapping from $\mathrm{L}^1(X)$ to $\mathbb{K}$.

So the real step in the dominated convergence theorem is the assertion that $\|f - f_n\|_1 \to 0$. A proof is in Exercise 7.28. □

Theorem 7.18 (Completeness of L^1). *The space $\mathrm{L}^1(X)$ is a Banach space. More precisely, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{L}^1(X)$. Then there are functions $f, g \in \mathrm{L}^1(X)$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that*

$$|f_{n_k}| \leq g \quad a.e. \quad and \quad f_{n_k} \to f \quad a.e..$$

Furthermore, $\|f_n - f\|_1 \to 0$.

Proof. Note first that if we have found f, g and the subsequence with the stated properties, then $\|f_{n_k} - f\|_1 \to 0$ by dominated convergence, and hence $\|f_n - f\|_1 \to 0$ since the sequence $(f_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_1$-Cauchy.

We find the subsequence in the following way. By using the Cauchy property we may pick $n_k < n_{k+1}$, $k \in \mathbb{N}$, such that $\left\|f_{n_k} - f_{n_{k+1}}\right\|_1 < \frac{1}{2^k}$. To facilitate notation let $g_k := f_{n_k}$. Then for every $N \in \mathbb{N}$,

$$\int_X \sum_{k=0}^{N} |g_k - g_{k+1}| \,\mathrm{d}\lambda = \sum_{k=0}^{N} \|g_k - g_{k+1}\|_1 \leq \sum_{k=0}^{\infty} \frac{1}{2^k} = 2.$$

Define $G := \sum_{k=0}^{\infty} |g_k - g_{k+1}|$ pointwise. Then by the monotone convergence theorem

$$\int_X G \,\mathrm{d}\lambda = \lim_{N\to\infty} \int_X \sum_{k=0}^{N} |g_k - g_{k+1}| \,\mathrm{d}\lambda \leq 2,$$

and hence $G \in \mathrm{L}^1(X)$. By Lemma 7.14, $\sum_{k=0}^{\infty} |g_k - g_{k+1}| = G < \infty$ a.e. Hence for almost all $x \in X$ the limit

$$h(x) := \sum_{k=0}^{\infty} g_k(x) - g_{k+1}(x) = g_0(x) - \lim_{n\to\infty} g_{n+1}(x)$$

exists. Hence $g_k \to f := g_0 - h$ almost everywhere, and

$$|g_k| \leq |g_0| + \sum_{j=0}^{k-1} |g_j - g_{j+1}| \leq |g_0| + G \quad \text{a.e.}$$

Thus, if we set $g := |g_0| + G$, the theorem is completely proved. □

Theorem 7.22 (Hölder's Inequality). *Let q be the dual exponent defined by $1/p + 1/q = 1$. If $f \in \mathrm{L}^p(X)$ and $g \in \mathrm{L}^q(X)$, then $fg \in \mathrm{L}^1(X)$ and*

$$\left| \int_X fg \, \mathrm{d}\lambda \right| \leq \|f\|_p \, \|g\|_q .$$

Proof. The case $p, q \in \{1, \infty\}$ has been treated above, so we shall suppose $1 < p, q < \infty$ in the following. The proof proceeds as the proof of Theorem 2.30. Recall from Lemma 2.31 the identity

$$ab = \inf_{t>0} \frac{t^p}{p} a^p + \frac{t^{-q}}{q} b^q$$

for real numbers $a, b \geq 0$. Inserting $a = |f(x)|, b = |g(x)|$ we obtain

$$|f(x)g(x)| \leq \frac{t^p}{p} |f(x)|^p + \frac{t^{-q}}{q} |g(x)|^q$$

for all $t > 0$ and all $x \in X$. Integrating yields

$$\int_X |fg| \, \mathrm{d}\lambda \leq \frac{t^p}{p} \int_X |f|^p \, \mathrm{d}\lambda + \frac{t^{-q}}{q} \int_X |g|^q \, \mathrm{d}\lambda$$

for all $t > 0$. Taking the infimum over $t > 0$ again yields

$$\begin{aligned} \int_X |fg| \, \mathrm{d}\lambda &\leq \inf_{t>0} \left[\frac{t^p}{p} \int_X |f|^p \, \mathrm{d}\lambda + \frac{t^{-q}}{q} \int_X |g|^q \, \mathrm{d}\lambda \right] \\ &= \left(\int_X |f|^p \, \mathrm{d}\lambda \right)^{1/p} \left(\int_X |g|^q \, \mathrm{d}\lambda \right)^{1/q} = \|f\|_p \, \|g\|_q . \end{aligned}$$

This shows that $fg \in \mathrm{L}^1(X)$ and concludes the proof. □

Density. Finally, we return to our starting point, namely the question of a natural "completion" of $\mathrm{C}[a,b]$ with respect to $\|\cdot\|_1$ or $\|\cdot\|_2$. If $X = [a,b]$ is a finite interval, $1 < p < \infty$ and $1/p + 1/q = 1$, then one has

$$\mathrm{C}[a,b] \subseteq \mathrm{C}_b(a,b) \subseteq \mathrm{L}^\infty(a,b) \subseteq \mathrm{L}^p(a,b) \subseteq \mathrm{L}^1(a,b)$$

with

$$\begin{aligned} \|f\|_1 &\le (b-a)^{1/q} \, \|f\|_p && \text{for all } f \in \mathrm{L}^p(a,b), \\ \|f\|_p &\le (b-a)^{1/p} \, \|f\|_{\mathrm{L}^\infty} && \text{for all } f \in \mathrm{L}^\infty(a,b), \\ \|f\|_\infty &= \qquad \|f\|_{\mathrm{L}^\infty} && \text{for all } f \in \mathrm{C}_\mathrm{b}(a,b). \end{aligned}$$

(The proof is an exercise.) The following result gives the desired answer to our question, but once again, we can do nothing but quote the result without being able to provide a proof here. Ex.7.15

Theorem 7.24. *The space* $\mathrm{C}[a,b]$ *is* $\|\cdot\|_p$*-dense in* $\mathrm{L}^p(a,b)$ *for* $1 \le p < \infty$.

Note: The space $\mathrm{C}_b(a,b)$ is *not* $\|\cdot\|_{\mathrm{L}^\infty}$-dense in $\mathrm{L}^\infty(a,b)$. Ex.7.16

Theorem 8.5. *Let H be an inner product space, and let $A \neq \emptyset$ be a complete convex subset of H. Furthermore, let $f \in H$. Then there is a unique vector $P_A f := g \in A$ with $\|f - g\| = d(f, A)$.*

Proof. Let us define $d := d(f, A) = \inf\{\|f - g\| \mid g \in A\}$. For $g, h \in A$ we have $\frac{1}{2}(g + h) \in A$ as A is convex. If both $h, g \in A$ minimize $\|\cdot - f\|$ we see from Figure 12 that $g = h$. Algebraically, we use the parallelogram identity

and compute

$$\begin{aligned} \|g - h\|^2 &= \|(g - f) - (h - f)\|^2 \\ &= 2\|g - f\|^2 + 2\|h - f\|^2 - 4\left\|\tfrac{1}{2}(g + h) - f\right\|^2 \\ &\leq 2\|g - f\|^2 + 2\|h - f\|^2 - 4d^2. \end{aligned}$$

Hence, if $\|g - f\|^2 = d^2 = \|h - f\|^2$ and we obtain

$$\|g - h\|^2 \leq 2d^2 + 2d^2 - 4d^2 = 0.$$

To show existence, let $(g_n)_{n\in\mathbb{N}}$ be a minimizing sequence in A, i.e., $g_n \in A$ and $d_n := \|f - g_n\| \searrow d$. For $m \geq n$ we replace g, h by g_n, g_m in the estimation above and obtain

$$\|g_n - g_m\|^2 \leq 2\|g_n - f\|^2 + 2\|g_m - f\|^2 - 4d^2 \leq 4(d_n^2 - d^2).$$

Since $d_n \to d$, also $d_n^2 \to d^2$. Therefore, $(g_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in A, and since A is complete, there is a limit $g := \lim_{n\to\infty} g_n \in A$. But the norm is continuous, and so

$$\|f - g\| = \lim_{n\to\infty} \|f - g_n\| = \lim_{n\to\infty} d_n = d,$$

and we have found our desired minimizer. □

Corollary 8.10 (Orthogonal Decomposition). *Let H be a Hilbert space, and let $F \subseteq H$ be a closed linear subspace. Then every vector $f \in H$ can be written in a unique way as $f = u + v$ where $u \in F$ and $v \in F^\perp$.*

Proof. Uniqueness: if $f = u + v = u' + v'$ with $u, u' \in F$ and $v, v' \in F^\perp$, then

$$u - u' = v' - v \in F \cap F^\perp = \{0\}$$

by the definiteness of the scalar product. Hence $u = u', v = v'$ as claimed. Existence: Simply set $u = P_F f$ and $v = f - P_F f$. □

Theorem 8.12 (Riesz–Fréchet[1]). *Let H be a Hilbert space and let $\varphi : H \to \mathbb{K}$ be a bounded linear functional on H. Then there exists a unique $g \in H$ such that*

$$\varphi(f) = \langle f, g \rangle \qquad \text{for all } f \in H.$$

Proof. Uniqueness: If $g, h \in H$ are such that $\langle f, g \rangle = \varphi(f) = \langle f, h \rangle$ for all $f \in H$, then

$$\langle f, g - h \rangle = \langle f, g \rangle - \langle f, h \rangle = \varphi(f) - \varphi(f) = 0 \qquad (f \in H).$$

Hence $g - h \perp H$ which is only possible if $g = h$.

Existence: If $\varphi = 0$, we can take $g := 0$, so we may suppose that $\varphi \neq 0$. In this case the closed linear subspace $\ker(\varphi)$ is not the whole space, so we may pick an orthogonal vector $h \perp \ker(\varphi)$ with $\|h\| = 1$. In particular, $\varphi(h) \neq 0$. Given $f \in H$ we hence have

$$h \perp f - \tfrac{\varphi(f)}{\varphi(h)} h, \quad \text{i.e,} \quad \langle f, h \rangle = \tfrac{\varphi(f)}{\varphi(h)}.$$

It follows that $\varphi(f) = \left\langle f, \overline{\varphi(h)} h \right\rangle$. Since f was arbitrary, we may take $g := \overline{\varphi(h)} h$ and are done. □

Note $h \perp f - \cdots$ follows from $v = \varphi(h) f - \varphi(f) h, \ \varphi(v) = 0$.

Then be careful with the inner product and you are OK.

Theorem 8.13. *Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of pairwise orthogonal elements of H. Consider the statements*

(i) *The series $f := \sum_{n=1}^{\infty} f_n$ converges in H.*

(ii) $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$.

Then (i) *implies* (ii) *and one has* **Parseval's identity**[2]

$$\|f\|^2 = \sum_{n=1}^{\infty} \|f_n\|^2 . \tag{8.1}$$

If H is a Hilbert space, then (ii) *implies* (i).

Only prove (i), (ii) optional

Proof. Write $s_n := \sum_{j=1}^{n} f_j$ for the partial sums. If $f = \lim_{n\to\infty} s_n$ exists in H, then by the continuity of the norm and Pythagoras one obtains

$$\begin{aligned} \|f\|^2 &= \left\| \lim_{m\to\infty} s_m \right\|^2 = \lim_{m\to\infty} \|s_m\|^2 = \lim_{m\to\infty} \left\| \sum_{j=1}^{m} f_j \right\|^2 \\ &= \lim_{m\to\infty} \sum_{j=1}^{m} \|f_j\|^2 = \sum_{j=1}^{\infty} \|f_j\|^2 . \end{aligned}$$

Since $\|f\| < \infty$, this implies (ii).

Conversely, suppose that (ii) holds and that H is a Hilbert space. Hence (i) holds if and only if the partial sums $(s_n)_{n\in\mathbb{N}}$ form a Cauchy sequence. If $m > n$, then by Pythagoras' theorem

$$\|s_m - s_n\|^2 = \left\| \sum_{j=n+1}^{m} f_j \right\|^2 = \sum_{j=n+1}^{m} \|f_j\|^2 \le \sum_{j=n+1}^{\infty} \|f_j\|^2 \to 0$$

as $n \to \infty$ by (ii), and this concludes the proof. □

IDEA: Use Pythagoras all the time!!!

Theorem 8.15. *Let H be a Hilbert space, let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal system in H, and let $f \in H$. Then one has* **Bessel's inequality**

$$\sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2 \le \|f\|^2 < \infty. \tag{8.2}$$

Moreover, the series

$$Pf := \sum_{j=1}^{\infty} \langle f, e_j\rangle\, e_j$$

is convergent in H, and $Pf = P_F f$ is the orthogonal projection of f onto the closed subspace

$$F := \overline{\operatorname{span}}\{e_j \mid j \in \mathbb{N}\}.$$

Finally, one has **Parseval's identity** $\|Pf\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2$.

Proof. For Bessel's inequality it suffices to establish the estimate

$$\sum_{j=1}^{n} |\langle f, e_j\rangle|^2 \le \|f\|^2$$

for arbitrary $n \in \mathbb{N}$. This is immediate from Lemma 1.10; see (1.1). By Bessel's inequality and the fact that H is complete (by assumption) Theorem 8.13 yields that the sum $Pf := \sum_{j=1}^{\infty} \langle f, e_j\rangle\, e_j$ is indeed convergent in H with Parseval's identity being true.

To see that $Pf = P_F f$ we only need to show that $Pf \in F$ and $f - Pf \perp F$. Since Pf is a limit of sums of vectors in F, and F is closed, $Pf \in F$. For the second condition, note that

$$\langle f - Pf, e_k\rangle = \langle f, e_k\rangle - \sum_{j=1}^{\infty} \langle f, e_j\rangle \langle e_j, e_k\rangle = \langle f, e_k\rangle - \langle f, e_k\rangle = 0$$

for every k. Hence $f - Pf \perp F$ by Corollary 4.14. □

From 1.10:

$$\|f\|^2 = \|(f - Pf) + Pf\|^2 = \|f - Pf\|^2 + \|Pf\|^2 \ge \|Pf\|^2$$

by Pythagoras' lemma. □

Lemma 9.17. *Let $f \in \mathrm{L}^1(a,b)$. Then*

$$\|f\|_1 = \sup\left\{\left|\int_a^b f(s)g(s)\,\mathrm{d}s\right| \mid g \in \mathrm{C}[a,b],\ \|g\|_\infty \le 1\right\}. \tag{9.2}$$

In particular, $f = 0$ a.e. if and only if $\int_a^b f(s)g(s)\,\mathrm{d}s = 0$ for all $g \in \mathrm{C}[a,b]$.

Only know by heart

Lemma 10.5. *The space* $\mathrm{L}^2(a,b)$ *decomposes orthogonally into*

$$\mathrm{L}^2(a,b) = \mathbb{C}\mathbf{1} \oplus \overline{\{\psi' \mid \psi \in \mathrm{C}_0^1[a,b]\}},$$

with $\|\cdot\|_2$*-closure on the right-hand side.*

Corollary 10.7. *One has* $\mathrm{H}^1(a,b) \subseteq \mathrm{C}[a,b]$*. More precisely,* $f \in \mathrm{H}^1(a,b)$ *if and only if* f *has a representation*

$$f = Jg + c\mathbf{1}$$

with $g \in \mathrm{L}^2(a,b)$ *and* $c \in \mathbb{K}$*. Such a representation is unique, namely*

$$g = f' \quad \text{and} \quad c = \frac{\langle f - Jf', \mathbf{1}\rangle}{b-a}.$$

Moreover, the **fundamental theorem of calculus** *holds, i.e.,*

$$\int_c^d f'(s)\,\mathrm{d}s = f(d) - f(c) \quad \text{for every interval } [c,d] \subseteq [a,b].$$

Only know by heart: this is how L^2 relates to derivatives, and how H^1 relates to L^2.

Lemma 10.10 (Poincaré Inequality[4]). *There is a constant* $C \geq 0$ *depending on* $b-a$ *such that*

$$\|u\|_{\mathrm{L}^2} \leq C\|u'\|_{\mathrm{L}^2} \tag{10.8}$$

for all $u \in \mathrm{H}_0^1(a,b)$*. In particular,* (10.7) *is an inner product and* $\|\cdot\|_{\mathrm{H}_0^1}$ *is a norm on* $\mathrm{H}_0^1(a,b)$*.*

Proof. Let $u \in \mathrm{H}_0^1(a,b)$. We claim that $u = Ju'$. Indeed, if $(Ju')' = u'$ and by Corollary 10.6 $Ju' - u = c$ is a constant. But $Ju' - u$ vanishes at a and hence $c = 0$. Using $Ju' = u$ we finally obtain

$$\|u\|_{\mathrm{L}^2} = \|Ju'\|_{\mathrm{L}^2} \leq C\|u'\|_{\mathrm{L}^2}$$

for some constant C, since by Lemma 10.4 the integration operator J is bounded on $\mathrm{L}^2(a,b)$. The remaining statements follow readily. □

Lemma 11.3. *Let $f \in \mathrm{L}^1(a,b)$ and $n \in \mathbb{N}$. Then*

$$(J^n f)(t) = \frac{1}{(n-1)!} \int_a^t (t-s)^{n-1} f(s)\,\mathrm{d}s \quad \text{for all } t \in [a,b].$$

In particular, J^n is again an integral operator, with kernel function

$$k_n(t,s) = \frac{1}{(n-1)!} \mathbf{1}_{[a,t]}(s)(t-s)^{n-1} \qquad (s,t \in [a,b]).$$

Proof. This is proved by induction and Fubini's theorem. □

Example 11.9 (Integration Operator)**.** The n-th power of the integration operator J on $E = \mathrm{C}[a,b]$ is induced by the integral kernel

$$k_n(t,s) = \mathbf{1}_{\{s \le t\}}(t,s) \frac{(t-s)^{n-1}}{(n-1)!}.$$

From this it follows that $\|J^n\|_{\mathcal{L}(E)} = 1/n! \neq 1^n = \|J\|^n$. (See Exercise 11.7.)

Only know by heart; Fubini only tells you how to integrate a function of 2 variables.

Definition 12.1. An operator $A : E \to F$ between normed spaces E and F is called of **finite rank** or a **finite-dimensional operator** if $\mathrm{ran}(A)$ has finite dimension. And it is called **finitely approximable** if there is a sequence $(A_n)_{n\in\mathbb{N}}$ of finite-dimensional operators in $\mathcal{L}(E;F)$ such that $\|A_n - A\| \to 0$.

Definition 12.5. A linear operator $A : E \to F$ between Banach spaces E, F is called **compact** if it has the property that whenever $(f_n)_{n\in\mathbb{N}}$ is a bounded sequence in E, then the sequence $(Af_n)_{n\in\mathbb{N}} \subseteq F$ has a convergent subsequence.

Corollary 12.10. *Let H, K be Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Then there is a unique bounded linear operator $A^* : K \to H$ such that*

$$\langle Af, g\rangle_K = \langle f, A^* g\rangle_H \quad \text{for all} \quad f \in H,\ g \in K.$$

Furthermore, one has $(A^*)^* = A$ and $\|A^*\| = \|A\|$.

Theorem 12.9. *Let H, K be Hilbert spaces, and let $b : H \times K \to \mathbb{K}$ be a sesquilinear form which is bounded, i.e., there is $c > 0$ such that*

$$|b(f,g)| \le c\, \|f\|\, \|g\| \quad \text{for all } f \in H,\ g \in K.$$

Then there is a unique linear operator $B : K \to H$ such that

$$b(f,g) = \langle f, Bg\rangle \quad \text{for all } f \in H,\ g \in K. \tag{12.2}$$

The operator B is bounded and $\|B\| \le c$.

Proof. Uniqueness: if B and B' are two operators with (12.2) then for each $g \in K$ we have $(Bg - B'g) \perp H$, i.e., $Bg - B'g = 0$. Hence, $B = B'$.

Existence: Fix $g \in K$. Then the mapping

$$\varphi : H \to \mathbb{K}, \qquad f \longmapsto b(f,g)$$

is a linear functional on H. By (12.2), φ is bounded, with $\|\varphi\| \le c\, \|g\|$. The Riesz–Fréchet theorem yields an element $h \in H$ that induces this functional, i.e., such that

$$b(f,g) = \varphi(f) = \langle f, h\rangle \quad \text{for all } f \in H.$$

The element h is unique with this property, and depends only on g, so we are allowed to write $Bg := h$. This defines $Bg \in H$ for each $g \in K$.

For the linearity of B we observe that for given $g, h \in K$ and $\lambda \in \mathbb{K}$,

$$\begin{aligned}\langle f, B(\lambda g + h)\rangle &= b(f, \lambda g + h) = b(f,g)\overline{\lambda} + b(f,h) \\ &= \langle f, Bg\rangle\, \overline{\lambda} + \langle f, Bh\rangle = \langle f, \lambda Bg + Bh\rangle\end{aligned}$$

for all $f \in H$, whence $B(\lambda g + h) = \lambda Bg + Bh$.

The boundedness of B follows from

$$\|Bg\| = \sup_{\|f\| \le 1} |\langle f, Bg\rangle| = \sup_{\|f\| \le 1} |b(f,g)| \le \sup_{\|f\| \le 1} c\, \|f\|\, \|g\| = c\, \|g\|$$

(cf. Example 2.23.) □

Proof. Only the last two statements have not been proved yet. Observe that

$$\langle g, (A^*)^* f\rangle = \langle A^* g, f\rangle = \overline{\langle f, A^* g\rangle} = \overline{\langle Af, g\rangle} = \langle g, Af\rangle$$

for all $f \in H$, $g \in K$. This implies that $(A^*)^* = A$. But then $\|A\| = \|(A^*)^*\| \le \|A^*\| \le \|A\|$ and the theorem is completely proved. □

Summary: try to apply Riesz-Fréchet for existence. Uniqueness is OK.

Definition 13.2. Let E be a normed space and $A : E \to E$ a bounded operator. A scalar $\lambda \in \mathbb{K}$ is called an **approximate eigenvalue** of A if there is a sequence $(f_n)_{n\in\mathbb{N}}$ in E such that $\|f_n\| = 1$ for all $n \in \mathbb{N}$ and $\|Af_n - \lambda f_n\| \to 0$.

Lemma 13.4. *Let A be a bounded operator on the Banach space E. If $\lambda\mathrm{I} - A$ is invertible, then λ cannot be an approximate eigenvalue. If $|\lambda| > \|A\|$, then $\lambda\mathrm{I} - A$ is invertible.*

Proof. If $\|Af_n - \lambda f_n\| \to 0$ and $\lambda\mathrm{I} - A$ is invertible, then

$$f_n = (\lambda\mathrm{I} - A)^{-1}(\lambda f_n - Af_n) \to 0$$

which contradicts the requirement that $\|f_n\| = 1$ for all $n \in \mathbb{N}$. Take $|\lambda| > \|A\|$. Then $\|\lambda^{-1}A\| < 1$ and hence

$$\lambda\mathrm{I} - A = \lambda(\mathrm{I} - \lambda^{-1}A)$$

is invertible with

$$(\lambda\mathrm{I} - A)^{-1} = \lambda^{-1}\sum\nolimits_{n=0}^{\infty}(\lambda^{-1}A)^n = \sum\nolimits_{n=1}^{\infty}\lambda^{-(n+1)}A^n$$

(Theorem 11.13). □

Theorem 11.13. *Let E be a Banach space and let $A \in \mathcal{L}(E)$ be such that*

$$\sum\nolimits_{n=0}^{\infty}\|A^n\| < \infty.$$

Then the operator $\mathrm{I} - A$ is invertible and its inverse is given by the so-called **Neumann series**[3]

$$(\mathrm{I} - A)^{-1} = \sum\nolimits_{n=0}^{\infty}A^n.$$

Theorem 13.8. *Let A be a bounded self-adjoint operator on a Hilbert space A. Then $\langle Af, f\rangle \in \mathbb{R}$ for all $f \in H$ and*

$$\|A\| = |||A||| := \sup\{|\langle Af, f\rangle| \mid f \in H, \|f\| = 1\}.$$

The quantity $|||A|||$ is called the **numerical radius** of A.

Proof. One has $\langle Af, f\rangle = \langle f, Af\rangle = \overline{\langle Af, f\rangle}$ so $\langle Af, f\rangle$ is real. By Cauchy–Schwarz,

$$|\langle Af, f\rangle| \leq \|Af\| \, \|f\| \leq \|A\| \, \|f\|^2 = \|A\| \, ,$$

if $\|f\| = 1$. This proves that $|||A||| \leq \|A\|$.

Lemma 13.10. *Let A be a compact self-adjoint operator on a Hilbert space. Then A has an eigenvalue λ such that $|\lambda| = \|A\|$.*

Only know by heart for the following theorem

Theorem 13.11 (Spectral Theorem). *Let A be a compact self-adjoint operator on a Hilbert space H. Then A is of the form*

$$Af = \sum_j \lambda_j \langle f, e_j \rangle e_j \qquad (f \in H) \tag{13.1}$$

for some (finite or countably infinite) orthonormal system $(e_j)_j$ and real numbers $\lambda_j \neq 0$ satisfying $\lim_{j\to\infty} \lambda_j = 0$. Moreover, $Ae_j = \lambda e_j$ for each j.

More precisely, the orthonormal system is either $(e_j)_{j=1}^N$ for some $N \in \mathbb{N}$ or $(e_j)_{j\in\mathbb{N}}$. Of course, the condition $\lim_{j\to\infty} \lambda_j = 0$ is only meaningful in the second case.

Proof. We shall find (e_j, λ_j) step by step. If $A = 0$, then there is nothing to show. So let us assume that $\|A\| > 0$.

Write $H_1 = H$. By Lemma 13.10, A has an eigenvalue λ_1 such that $|\lambda_1| = \|A\|$. Let $e_1 \in H$ be such that $\|e_1\| = 1$ and $Ae_1 = \lambda_1 e_1$.

Now $F_1 := \operatorname{span}\{e_1\}$ is clearly an A-invariant linear subspace of H_1. By Lemma 13.9.c), $H_2 := F_1^\perp$ is also A-invariant. Hence we can consider the restriction $A|H_2$ of A on H_2 and iterate. If $A|H_2 = 0$, the process stops. If not, since $A|H_2$ is a compact self-adjoint operator on H_2, we can find a unit vector e_2 and a scalar λ_2 such that $Ae_2 = \lambda_2 e_2$ and

$$|\lambda_2| = \|A|H_2\|_{\mathcal{L}(H_2)} \leq \|A|H_1\|_{\mathcal{L}(H_1)} .$$

After n steps we have constructed an orthonormal system $e_1, \dots, e_n$ and a sequence $\lambda_1, \dots, \lambda_n$ such that

$$Ae_j = \lambda_j e_j, \quad |\lambda_j| = \|A|H_j\|_{\mathcal{L}(H_j)} \quad \text{where} \quad H_j = \{e_1, \dots, e_{j-1}\}^\perp$$

for all $j = 1, \dots, n$. In the next step define $H_{n+1} := \{e_1, \dots, e_n\}^\perp$, note that it is A-invariant and consider the restriction $A|H_{n+1}$ thereon. This is again a compact self-adjoint operator. If $A|H_{n+1} = 0$, the process stops, otherwise one can find a unit eigenvector associated with an eigenvalue λ_{n+1} such that $|\lambda_{n+1}| = \|A|H_{n+1}\|_{\mathcal{L}(H_{n+1})}$.

Suppose that the process stops after the n-th step. Then $A|H_{n+1} = 0$. If $f \in H$, then

$$f - \sum_{j=1}^n \langle f, e_j \rangle e_j \in \{e_1, \dots, e_n\}^\perp = H_{n+1}$$

and so A maps it to 0; this means that

$$Af = A \sum_{j=1}^n \langle f, e_j \rangle e_j = \sum_{j=1}^n \langle f, e_j \rangle Ae_j = \sum_{j=1}^n \lambda_j \langle f, e_j \rangle e_j,$$

i.e., (13.1). Now suppose that the process does not stop, i.e., $|\lambda_n| = \|A|H_n\| > 0$ for all $n \in \mathbb{N}$. We claim that $|\lambda_n| \to 0$, and suppose towards a contradiction that this is not the case. Then there is $\epsilon > 0$ such that $|\lambda_n| \geq \epsilon$ for all $n \in \mathbb{N}$. But then

$$\|Ae_j - Ae_k\|^2 = \|\lambda_j e_j - \lambda_k e_k\|^2 = |\lambda_j|^2 + |\lambda_k|^2 \geq 2\epsilon^2$$

for all $j \neq k$. So $(Ae_j)_{j\in\mathbb{N}}$ cannot have a convergent subsequence, contradicting the compactness of A.

Now let $f \in H$ and define

$$y_n := f - \sum_{j=1}^{n-1} \langle f, e_j \rangle e_j \ \in \ \{e_1, \dots, e_{n-1}\}^\perp = H_n.$$

Note that y_n is the orthogonal projection of f onto H_n, and so $\|y_n\| \leq \|f\|$. Hence

$$\|Ay_n\| \leq \|A|H_n\|_{\mathcal{L}(H_n)} \|y_n\| \leq |\lambda_n| \|f\| \to 0.$$

This implies

$$Af - \sum_{j=1}^{n-1} \lambda_j \langle f, e_j \rangle e_j = Ay_n \to 0,$$

which proves (13.1). □

Proof idea: keep picking $|\lambda_i| = \|A|H_1\|$ with corresponding eigenvectors.

Theorem 15.1 (Baire). *Let (Ω, d) be a nonempty complete metric space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of Ω such that*

$$\Omega = \bigcup\nolimits_{n\in\mathbb{N}} A_n.$$

Then there is $n \in \mathbb{N}$ and $x \in \Omega, r > 0$ with $\mathrm{B}(x, r) \subseteq A_n$.

May use the following:

Lemma 15.2 (Principle of Nested Balls). *Let (Ω, d) be a complete metric space, and let*

$$\mathrm{B}[x_1, r_1] \supseteq \mathrm{B}[x_2, r_2] \supseteq \mathrm{B}[x_3, r_3] \supseteq \dots$$

be a nested *sequence of closed balls in it. If $r_n \to 0$, then $x := \lim_{n\to\infty} x_n$ exists and*

$$\bigcap\nolimits_{n\in\mathbb{N}} \mathrm{B}[x_n, r_n] = \{x\}. \tag{15.1}$$

Proof of Theorem 15.1. We suppose that no A_n contains an open ball, and claim that then there exists $x \in \Omega$ which is not contained in any A_n. To find that x we are going to construct a sequence of nested balls.

In the first step, pick any $x_1 \in \Omega \setminus A_1$. This must exist, otherwise $A_1 = \Omega$, which trivially contains an open ball. Since A_1 is closed, if $r_1 > 0$ is small enough one has

$$A_1 \cap \mathrm{B}[x_1, r_1] = \emptyset.$$

By hypothesis, A_2 does not contain $\mathrm{B}(x_1, r_1)$, so there is $x_2 \in \mathrm{B}(x_1, r_1)$ but $x_2 \notin A_2$. Since A_2 is closed and $\mathrm{B}(x_1, r_1)$ is open, for $r_2 > 0$ small enough we have

$$A_2 \cap \mathrm{B}[x_2, r_2] = \emptyset \quad \text{and} \quad \mathrm{B}[x_2, r_2] \subseteq \mathrm{B}(x_1, r_1).$$

Again by hypothesis, the set A_3 does not contain the ball $\mathrm{B}(x_2, r_2)$, and hence we find $x_3 \in \mathrm{B}(x_2, r_2)$ but $x_3 \notin A_3$. Since A_3 is closed and $\mathrm{B}(x_2, r_2)$ is open, for small enough $r_3 > 0$ we have

$$A_3 \cap \mathrm{B}[x_3, r_3] = \emptyset \quad \text{and} \quad \mathrm{B}[x_3, r_3] \subseteq \mathrm{B}(x_2, r_2).$$

In this manner we find a nested sequence of balls $\mathrm{B}[x_n, r_n]$ such that

$$\mathrm{B}[x_n, r_n] \cap A_n = \emptyset \qquad \text{for all } n \in \mathbb{N}. \tag{15.2}$$

Since in each step we can make the radius r_n as small as we like, we can arrange it such that $r_n \to 0$. By the principle of nested balls, the centers $(x_n)_{n\in\mathbb{N}}$ converge to $x \in \bigcap_{n\in\mathbb{N}} \mathrm{B}[x_n, r_n]$. By (15.2), $x \notin A_n$ for each $n \in \mathbb{N}$, and the proof is complete. ☐

Definition 15.3. Let E, F be normed linear spaces. A collection $\mathcal{T}$ of linear mappings from E to F is called **uniformly bounded** if there is a $c \geq 0$ such that

$$\|Tf\| \leq c\,\|f\| \qquad \text{for all } f \in E \text{ and all } T \in \mathcal{T}.$$

In other words, $\mathcal{T}$ is uniformly bounded if each $T \in \mathcal{T}$ is bounded and

$$\sup\{\|T\| \ \mid T \in \mathcal{T}\} < \infty,$$

i.e., $\mathcal{T}$ is a bounded subset of the normed space $\mathcal{L}(E;F)$.

Suppose that $\mathcal{T} \subseteq \mathcal{L}(E;F)$ is a uniformly bounded collection of linear operators. Then for each $f \in E$ one has

$$\|Tf\| \leq \|T\|\,\|f\| \leq \big(\sup_{S\in\mathcal{T}} \|S\|\big)\ \|f\|$$

for all $T \in \mathcal{T}$, and hence $\sup_{T\in\mathcal{T}} \|Tf\| < \infty$. We say that the operator family $\mathcal{T}$ is *pointwise bounded*. The uniform boundedness principle asserts that in case that E is complete, i.e., a Banach space, one has the converse implication.

Theorem 15.4 (Uniform Boundedness Principle). *Let E be a Banach space, let F be a normed space, and let $\mathcal{T}$ be a collection of bounded linear operators from E to F. Then $\mathcal{T}$ is uniformly bounded if and only if it is pointwise bounded.*

May use the following Lemma

Lemma 15.5. *Let $(E, \|\cdot\|)$ be a normed space and let $K \subseteq E$ be a subset with the following properties:*

1) *K is "midpoint-convex", i.e., if $f, g \in K$, then also $\frac{1}{2}(f+g) \in K$;*
2) *K is "symmetric", i.e., if $f \in K$, then also $-f \in K$.*

Then, if K contains some open ball of radius $r > 0$, it also contains $\mathrm{B}(0, r)$.

Proof of Theorem 15.4. Let $\mathcal{T} \subseteq \mathcal{L}(E; F)$ be pointwise bounded. For $n \in \mathbb{N}$, define

$$K_n := \{f \in E \mid \|Tf\| \le n \text{ for all } T \in \mathcal{T}\}.$$

Since each $T \in \mathcal{T}$ is bounded and the norm mapping is continuous, K_n is a closed subset of E. By hypothesis, every $f \in E$ is contained in at least one K_n, so

$$E = \bigcup\nolimits_{n \in \mathbb{N}} K_n.$$

Since E is complete, Baire's theorem applies and yields $n \in \mathbb{N}$, $r > 0$, and $g \in E$ with $\mathrm{B}(g, r) \subseteq K_n$. By straightforward arguments, K_n is midpoint-convex and symmetric. Hence Lemma 15.5 implies that $\mathrm{B}(0, r) \subseteq K_n$, and since K_n is closed, we have even $\mathrm{B}[0, r] \subseteq K_n$, by Exercise 15.1.

Now take $f \in E$ with $\|f\| \le 1$. Then $rf \in \mathrm{B}[0, r] \subseteq K_n$, which means that $r\|Tf\| = \|T(rf)\| \le n$ for each $T \in \mathcal{T}$. Dividing by r yields

$$\|Tf\| \le n/r \qquad \text{for all } T \in \mathcal{T},$$

and hence $\|T\| \le n/r$ for all $T \in \mathcal{T}$. □

Theorem 15.6 (Banach–Steinhaus[2]). *Let E, F be Banach spaces, and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$ be a sequence such that*

$$Tf := \lim_{n\to\infty} T_n f$$

exists for every $f \in E$. Then T is a bounded operator, $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, and

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\| .$$

Proof. For each $f \in E$, since $(T_n f)_{n\in\mathbb{N}}$ converges, also $(\|T_n f\|)_{n\in\mathbb{N}}$ converges, and therefore $\sup_{n\in\mathbb{N}} \|T_n f\| < \infty$. By the uniform boundedness principle, $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. If $f \in E$ with $\|f\| \leq 1$, then by the continuity of the norm,

$$\|Tf\| = \lim_{n\to\infty} \|T_n f\| = \liminf_{n\to\infty} \|T_n f\| \leq \liminf_{n\to\infty} \|T_n\| .$$

Taking the supremum over all such f concludes the proof. □

Remember as limit of (T_n) bounded linear functionals exists pointwise -> T is bnd and (T_n) uniformly.

Remember proof as: try to go to uniform boundedness principle.

Theorem 15.8 (Open Mapping Theorem). *Let E, F be Banach spaces and let $T : E \to F$ be a bounded linear mapping which is surjective. Then there is $a > 0$ such that for each $g \in F$ there is $f \in E$ with $\|f\| \le a \|g\|$ and $Tf = g$.*

Onto. Remember as closeness of the inverse, in so far as it exists.

May use "Rough surjective + approx. pre-images" = surjective.

Theorem 15.11. *Let E, F be Banach spaces, and let $T \in \mathcal{L}(E; F)$. Suppose that there exist $0 \le q < 1$ and $a \ge 0$ such that for every $g \in F$ with $\|g\| \le 1$ there is $f \in E$ such that*

$$\|f\| \le a \quad \textit{and} \quad \|Tf - g\| \le q.$$

Then for each $g \in F$ there is $f \in E$ such that $Tf = g$ and $\|f\| \le \frac{a}{1-q} \|g\|$.

Then

Proof of Theorem 15.8. Let E, F be Banach spaces and let $T \in \mathcal{L}(E; F)$ be surjective. For $n \in \mathbb{N}$ define

$$B_n := \{g \in F \mid \exists f \in E \text{ s.t. } \|f\| \le n,\ Tf = g\} = T(\mathrm{B}_E[0, n]).$$

Note that B_n is midpoint-convex and symmetric, hence — by Exercise 15.3 — $A_n := \overline{B_n}$ has the same properties. Moreover, $F = \bigcup_{n \in \mathbb{N}} B_n$, by the surjectivity of T, and hence

$$F = \bigcup\nolimits_{n \in \mathbb{N}} A_n.$$

Since all the A_n are closed, and F is complete, Baire's theorem applies and we find an $n \in \mathbb{N}$ such that A_n contains an open ball. But A_n is midpoint-convex and symmetric, whence by Lemma 15.5 there is $r > 0$ such that $\mathrm{B}_E(0, r) \subseteq A_n$. Since A_n is closed, we have even

$$\mathrm{B}_F[0, r] \subseteq A_n = \overline{B_n} = \overline{T(\mathrm{B}_E[0, n])}. \tag{15.4}$$

Dividing by r yields

$$\mathrm{B}_F[0, 1] \subseteq \overline{T(\mathrm{B}_E[0, n/r])},$$

and this means that the hypotheses of Theorem 15.11 are satisfied with $a = n/r$ and any $q \in (0, 1)$. The conclusion of Theorem 15.11 yields exactly what we want. □

May use relatively more intuitive explanation, which skips Lemma 15.5.

Def: graph(T)

The Closed Graph Theorem. Let E, F be two normed spaces. A linear mapping $T : E \to F$ is said to have a **closed graph** if

$$f_n \to f, \quad Tf_n \to g \quad \Longrightarrow \quad Tf = g$$

holds for all sequences $(f_n)_{n\in\mathbb{N}}$ in E and all $f \in E$, $g \in F$. That means, T has closed graph if its graph

$$\text{graph}(T) = \{(f, Tf) \mid f \in E\}$$

is closed in the normed space $E \times F$; see Exercise 4.20.

Theorem 15.12 (Closed Graph Theorem). *If E, F are Banach spaces and $T : E \to F$ is a linear mapping, then T is bounded if and only if it has a closed graph.*

Proof uses following cor, also to be proven:

Corollary 15.10. *Let E be a linear space that is a Banach space with respect to either one of two given norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on E. If there is $M \geq 0$ with*

$$\|f\|_2 \leq M\,\|f\|_1 \qquad \textit{for all } f \in E,$$

then the two norms are equivalent.

Proof. The hypothesis just says that $\mathrm{I} : (E, \|\cdot\|_1) \to (E, \|\cdot\|_2)$ is bounded. If both are Banach spaces, Corollary 15.9 applies, and hence $\mathrm{I} = \mathrm{I}^{-1} : (E, \|\cdot\|_2) \to (E, \|\cdot\|_2)$ is bounded, too. This yields a constant $M' \geq 0$ such that $\|f\|_1 \leq M'\,\|f\|_1$ for every $f \in E$, and hence both norms are equivalent. □

Uses 15.9, which says that if T is 1-1 & onto, $\exists T^{-1}$ which is bounded by some other M'.

Proof. Define the new norm $\|\!| f |\!\| := \|f\|_E + \|Tf\|_F$ for $f \in E$. Then $\|f\|_E \leq \|\!| f |\!\|$ for each $f \in E$. The closedness of the graph of T and since both E, F are complete implies that E is complete with respect to this new norm. Hence by Corollary 15.10 there must be a constant $c > 0$ such that $\|Tf\| \leq \|\!| f |\!\| \leq c\,\|f\|$ for all $f \in E$. □

Other way is easy.

Definition of seperable normed space: if it contains a fundamental set, $\overline{span}\{f_n\} = E$.
This should be *countable*.

Theorem 16.2 (Hahn–Banach,[1] Separable Case). *Let E be a separable normed space over the scalar field $\mathbb{K}$. Let $E_0 \subseteq E$ be a subspace and $\varphi_0 \in E_0'$ a bounded linear functional on E_0. Then there is an extension $\varphi \in E'$ of φ_0 to all of E with $\|\varphi\| = \|\varphi_0\|$.*

May use

Lemma 16.1. *Let E be a real linear space and let $p : E \to \mathbb{R}$ be a sublinear functional. Furthermore, let $F \subseteq E$ be a linear subspace, $\varphi : F \to \mathbb{R}$ a linear mapping with $\varphi \le p$ on F. Given any $h \in E \setminus F$ there is $\alpha \in \mathbb{R}$ such that the definition*

$$F_1 := F \oplus \mathbb{R}\, h, \quad \varphi_1(f + th) := \varphi(f) + \alpha\, t \qquad (t \in \mathbb{R},\ f \in F)$$

yields a linear mapping $\varphi_1 : F_1 \to \mathbb{R}$ with $\varphi_1|_F = \varphi$ and $\varphi_1 \le p$ on F_1.

And

Theorem 9.28 (Extension Theorem). *Let E be a normed space and let $E_0 \subseteq E$ be a dense subspace. Furthermore, let $T_0 : E_0 \to F$ be a bounded linear operator into a Banach space F. Then T_0 extends uniquely to a bounded linear operator $T : E \to F$. Moreover, T has the same norm as T_0, i.e.,*

$$\|T\|_{E \to F} = \|T_0\|_{D \to F} \,.$$

Proof. For the proof we first suppose that $\mathbb{K} = \mathbb{R}$ and (without loss of generality) $\|\varphi_0\| = 1$. Then define the sublinear functional $p : E \to \mathbb{R}$ by $p(f) := \|f\|$ for $f \in E$. Then $\varphi_0 \le p$ on E_0.

Since E is separable we can find a countable fundamental set $\{f_n \mid n \in \mathbb{N}\}$. Define $E_n := E_0 + \operatorname{span}\{f_1, \dots, f_n\}$ for $n \in \mathbb{N}$ and obtain an increasing chain of subspaces

$$E_0 \subseteq E_1 \subseteq E_2 \cdots \subseteq E.$$

Passing from E_n to E_{n+1} either nothing changes or the dimension increases by one. By Lemma 16.1 we can extend φ_0 stepwise along this chain of subspaces to obtain a linear functional

$$\varphi_\infty : E_\infty \to \mathbb{R} \quad \text{on} \quad E_\infty := E_0 + \operatorname{span}\{f_1, f_2, \dots\} = \bigcup\nolimits_{n\in\mathbb{N}} E_n$$

extending φ_0 and satisfying $\varphi \le p$ on E_∞. But this means that

$$|\varphi_\infty(f)| \le \|f\| \quad \text{for all } f \in E_\infty,$$

i.e., φ_∞ is bounded with norm ≤ 1. The extension theorem (Theorem 9.28) then yields an extension of φ_∞ to a bounded linear functional $\varphi \in E'$ with the same norm, and that is what we were aiming at.

Examples

Examples Counter-examples

Compactness	
Any closed subset of a compact space	The closed unit ball in l^2
Closed unit interval of real numbers.	In fact any infinite dimensional inner product space has this: take in it any sequence that is pairwise orthogonal as a counter-example.
Complete	
L^2 with proper norm	$C[a,b]$ with the standard inner product
	Any normed space with countable algebraic basis
Finite rank operators	
Any bounded linear functional has rank 1	Integration operator (generally)
Finitely approximable operators	
Hilbert-Schmidth integral operators	
Compact operators	
Any finitely approximable operator from a Banach space to a Hilbert space.	
Adjoint operators, $A \sim A^*$ s.t. $< Ax, y > = < x, A^* y >$	
Any matrix with its transpose	
The right and left – shift operators on l^2	
Operator with eigenvalues	
Identity operator has $\lambda = 1$	Multiplication operator: $(Af)(t) \coloneqq t \cdot f(t)$
Operator with approximate eigenvalues	
Multiplication operator	
Self-adjoint operators $A = A^*$	
Every orthogonal projection	
Every Hilbert-Schmidt integral operator, if we have $\overline{k(x,y)} = k(y,x)\ \ for\ \ a.e.\ \ x,y \in (a,b)$	
Of an application of the Spectral theorem for self-adjoint operators	
It helps to describe nasty differential equations. Especially useful in showing convergence & existence of solutions	
Seperable spaces (they have a countable fundamental set)	
l^p	l^∞: consider the set of {0,1} sequences, which is an uncountable discrete metric space.