AFA EPIC CHEAT SHEET

Mooie word cover pagina

AFA is big enough as it is

We have a hundred spaces, norms on these spaces, and operators. I don’t want to learn them all by heart and I don’t want to read through the entire book to find the correct definitions.

So I present these pages in which they can be concisely found!
Ignore the ugly page numbering

Erik Leering
ferikmaster@gmail.com

Introductory note

The central topics of AFA to my understanding are that of completeness of a certain normed space and boundedness of operators.

Completeness is of central importance because of the nice properties that follow from it. To this end we even go as far as to define a new integral using the techniques of **Lebesque**. In particular, Hilbert spaces have nice properties. From CH8, letting H be a Hilbert space:

- $F \subseteq H$ a closed linear subspace $\rightarrow \exists! P_F f \in F$ a best approximation of $f \in H$.
- $\boldsymbol{Riesz - Fréchet}$: each functional $\varphi \in \mathcal{L}(H;\mathbb{K})$ can uniquely be described by one element $g \in H$: $\varphi(f) = < f, g >_H \quad \forall f \in H$.
- Other theorems such as **Parseval, Bessel & Parseval's identity** hold on Hilbert spaces.
- **By the appendix F, every Hilbert space contains a (possibly uncountable) maximal ONS.** Such a maximal ONS ~ ONB, nearly fully describes the Hilbert space (CH 8 properties). If the Hilbert space is separable, then every orthonormal system in it is at most countable!

On completeness & Completion

Incompleteness of a metric space is not a substantial problem, because every metric space can be viewed as a dense subset of a complete metric space. Such a "surrounding" space is called a **completion**, and it can be constructed by the same methods that Cantor[1] and Heine[2] used to construct the real numbers from the rationals; see Appendix B. For normed spaces there is a more explicit construction (Corollary 16.11).

Corollary 16.11. *Every normed space is isometrically isomorphic to a dense subspace of a Banach space.*

Proof. By Theorem 11.8 each dual space is complete, i.e., a Banach space. The linear isometry $\mathrm{j} : E \to E''$ then maps E to the subspace $\mathrm{j}(E)$ of the Banach space E'', and hence $\overline{\mathrm{j}(E)}$ is a Banach space which has $\mathrm{j}(E)$ as a dense subspace. □

I strongly recommend having at least a look at Appendix B to find some intuition on this idea. It would have been nice if this material was included formally as it does give some good insight about the nature of "completeness" (or rather, the absence of it).

Bounded linear mappings/operators

Are relevant for their convenient properties as well, many of which are obvious. Riesz-Fréchet only works on bounded linear mappings. Many of the useful operators that we encounter (of which many are listed in this summary), are bounded. Boundedness plays an essential role in solving certain differential equations (chapters 10-11). For such operators, we can speak of (approximate) eigenvalues. The spectral theorem and its consequences apply to special bounded operators.

Also, the uniform boundedness principle and the open mapping theorem concern bounded operators. This should convince you of their usefulness.

Many quick/specific definitions are dropped in this page. For definition of certain spaces continue below.

Afa Definitions

Space structure	Complete name
$(\Omega, <.,.>)$ inner pr	Hilbert
$(\Omega, ||.||)$ Normed	Banach
$(\Omega, d(.,.))$ metric	Complete

Structural properties

Inner product:

Definition 1.2. Let E be a vector space. A mapping

$$E \times E \longrightarrow \mathbb{K}, \quad (f,g) \longmapsto \langle f,g \rangle$$

is called an **inner product** or a **scalar product** if it is *sesquilinear*:

$$\langle \lambda f + \mu g, h \rangle = \lambda \langle f,h \rangle + \mu \langle g,h \rangle,$$
$$\langle h, \lambda f + \mu g \rangle = \overline{\lambda} \langle h,f \rangle + \overline{\mu} \langle h,g \rangle \qquad (f,g,h \in E,\ \lambda,\mu \in \mathbb{K}),$$

symmetric:

$$\langle f,g \rangle = \overline{\langle g,f \rangle} \qquad (f,g \in E),$$

positive:

$$\langle f,f \rangle \geq 0 \qquad (f \in E),$$

and *definite*:

$$\langle f,f \rangle = 0 \implies f = 0 \qquad (f \in E).$$

Norm:

Definition 2.5. Let E be a vector space over the field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. mapping

$$\|\cdot\| : E \longrightarrow \mathbb{R}_+$$

is called a **norm** on E if it has the following properties:

1) $\|f\| = 0 \iff f = 0$ for all $f \in E$ (definiteness)
2) $\|\lambda f\| = |\lambda| \|f\|$ for all $f \in E, \lambda \in \mathbb{K}$ (homogeneity)
3) $\|f+g\| \leq \|f\| + \|g\|$ for all $f,g \in E$ (triangle inequality)

Operator definitions, for T

Bounded $\exists c \geq 0\ s.t.\ ||Tf|| \leq c||f||\, \forall f \in E.$

Operator norm $||T|| := sup_{||f|| \leq 1} ||Tf||$

Finite $\boldsymbol{rank(T)} := \dim(range(T)) < \infty$

Finitely approximable if $\exists (T_n) \in \mathcal{L}(E;F)$ all of finite rank such that $||T_n - T|| \to 0.$

Compact operator (f_n) *bounded in E* $\to$ $(Tf_n) \subseteq F$ has a convergent subsequence.

Definitions

CENTRAL: Complete: every Cauchy sequences converges to an element in the space.

$\boldsymbol{(f_n)}$ **is Cauchy** if $\forall \epsilon > 0, \exists N \in \mathbb{N}\ s.t.\ n,m > N \to ||f_n - f_m|| < \epsilon.$

Open set if O: $\forall x \in O\ \exists \epsilon > 0: B(x,\epsilon) \subseteq O.$

Closed set $A = \bar{A} \subseteq \Omega$
if $x_n \in A\ \ s.t.\ x_n \to x \in \Omega \Rightarrow x_n \to x \in A.$

Closure $\boldsymbol{\bar{A}} := \{x \in \Omega \mid \exists x_n \in A\ s.t.\ x_n \to x\}$

Compact $\boldsymbol{(\Omega, d)}$ is when every sequence in the set has a convergent subsequence.

Compact support: when a function f vanishes outside a finite interval (a,b).

Convex: $f, g \in A, t \in [0,1]$
$\to tf + (1-t)g \in A$

A is Lebesque Measure is $\lambda^*(A) := \inf \Sigma |Q_n|$ with Q_n a sequence of intervals that cover A. If the Lebesque measure is finite, the set is **Lebesque measurable.**

$\boldsymbol{f \in A}$ **is Lebesque measurable** if $\{t \in A \mid a \leq f(t) \leq b\} \in \Sigma\ \forall a,b \in \mathbb{R}.$

Isometry: $||Tf||_F = ||f||_E\ \forall f.$
Always 1-1, if also onto, then **isometric isomorphism.**

A normed space is **seperable** if $\exists A \subseteq E$ countable s.t. $\overline{span}(A) = E.$
e.g. $l^2(\mathbb{N})$ *is seperable with* $A = \{e_n\}.$
l^∞ *is not.*

Spaces of note

For below: I made this to be read left to right, to highlight symmetries. Does not mean *everything* placed next to each other has symmetry, but you will see the places where it obviously does have symmetry.

Norms: D_p = discrete p-norm $(\Sigma|x_n|^p)^{1/p}$, C_p = integral p-norm, $\left(\int |f|^p ds\right)^{1/p}$. OP $= ||T|| = \inf_{||f||\leq 1} ||Tf||$

Space	Norm		Space	Norm	
$c := \{(x_n) \subseteq \mathbb{K} \mid x_n \to x \in \mathbb{K}\}$	$D_{p=1,p,\infty}$	Y Y	$C(E;F) := \{A \in \mathcal{L}(E;F) \mid A \text{ is a compact operator}\}$	OP	
$c_0 := \{(x_n) \mid x_n \to 0\}$		Y	$C_0(E;F) := \{A \in \mathcal{L}(E;F) \mid A \text{ is finitely approximable}\}$	OP	
$c_{00} := \{(x_n) \mid x_n = 0, \text{eventually}\}$	$D_{p=1,p,\infty}$	N N N	$C_{00}(E;F) := \{A \in \mathcal{L}(E;F) \mid A \text{ has finite rank}\}$	OP	
$\mathcal{B}(\Omega) :=$ *all bounded functions on* Ω Note not necessarily continuous. However, this kind of proves completeness for the ones below, due to closedness of them.	C_1 C_p C_∞	N N Y			
$C :=$ the set of continuous functions	C_1 C_p C_∞	N N Y			
$C^1 :=$ the set of once cont. diff. functions.					
$C_b := \{f \in \mathcal{B}(\Omega) \mid f \text{ is continuous}\}$					
$C_0 := \{f \text{ cont} \mid f(a) = f(b) = 0\}$ Or continuous functions with compact support.					
$C_{per} := \{f \text{ cont} \mid f(0) = f(1)\}$					
$C_0^1 := C_0 \cap C^1$	C_1 C_p C_∞	N N N	(For below: let X be any interval, and) $\Sigma\ (\subseteq P(\mathbb{R}))$:=the set of all Lebesque-measurable sets Is the set on which we restrict ourselves.	---	---
			$M(X) := \{f: X \to \mathbb{K} \mid f \text{ is Lebesque} - \text{Measurable}\}$	---	---
			$\mathcal{L}^1 := \{f \in M(X) : \|f\|_1 < \infty\}$	---	---
$l_1 := \{(x_n) \mid \Sigma|x_n| < \infty\}$	D_1	Y	$L_1 := \mathcal{L}^1/{\sim\lambda}$ with equivalence relation for almost everywhere	C_1	Y
$l_2 := \{(x_n) \mid (\Sigma|x_n|^2)^{1/2} < \infty\}$	D_2	Y	$L_2 :=$ “”	C_2	Y
$l_p := \{(x_n) \mid (\Sigma|x_n|^p)^{1/p} < \infty\}$	D_p	Y	$L_p :=$ “”	C_p	Y
$l_\infty := \{(x_n) \mid \sup|x_n| < \infty\}$	D_∞	Y	$L_\infty :=$“” with $\|f\|_{L^\infty} := \inf\{c \geq 0 \mid |f| \leq c,\ a.e.\}$	$C\infty$	Y
			$BV([a,b];E) := \{f: [a,b] \to E \mid \|f\|_v < \infty\}$ BND variation $\|f\|_v = sup \sum \|f(t_j) - f(t_{j-1})\|_{E'}$ sup over all partitions.		
			$St([a,b];E) := \{f: [a,b] \to E \mid \exists \text{ partitioning } (t_n) \text{ of } [a,b]$ $s.t.\ \ f(t) = x_n \text{ on } [t_{n-1}, t_n]\}$	C_∞	N
			$Reg([a,b];E) := \overline{St([a,b];E)}$ in $\mathcal{B}([a,b];E)$ w.r.t $||.||_\infty$	C_∞	Y
$\mathcal{L}(E;F)$:=the space of bounded linear functions from E to F.	OP, when F is Banach	Y	$E' := \mathcal{L}(E;\mathbb{K})$ the dual space to normed space E	OP by left side	Y
$H^1 := \{f \in L^2 \mid \exists a \text{ weak derivative } f'\}$ $< f,g >_{H^1} = < f,g >_{L^2} + < f',g' >_{L^2}$	$\|f\|_{H^1}$ $= \left(\|f\|_2^2 + \|f'\|_2^2\right)^{1/2}$	Y	$L_c^p := \{f \in L^p,\ f \text{ has compact support}\}$	C_p	Y
$H_0^1 := H^1 \cap C_0$ $< f,g >_{H_0^1} = < f',g' >_{L^2}$	$\|f\|_{H^1}$ OR $\|f\|_{H_0^1} = \|f'\|_2$	Y Y			
$H^p := \{f \in H^1 \mid f' \in H^{p-1}\}$ $< f,g >_{H^p} = \Sigma_{k=0}^p < f^{(k)}, g^{(k)} >_{L^2}$	$\|f\|_{H^p}^2 = \Sigma \|f^{(k)}\|_2^2$	Y			
$H_0^2 =: dom(\Delta_D) = dom(L)$	Same as above for p=2 Note: closed subspace of $H^2(a,b)$, therefore→	Y			

Baire theorem consequences:

THM: A normed space with a countable algebraic basis is never complete. Note that **countable** implies infinite elements. So in this context, R^3 is complete, since it has a finite, nut a countable, basis.

On scales, strong and weak norms

$l^1 \subseteq l^2 \subseteq l^\infty$ In fact all of these are **strict**. (ex 3.4)

On finite dimensional (linear) spaces, all norms are equivalent (because they are all equivalent to the euclidean 2-norm on K^d, and there is an isometric isomorphism from E to K^d).

$L^\infty(a,b) \subseteq L^p(a,b) \subseteq L^1(a,b)$. All of these are proper inclusions.
None of these hold if we replace intervals with R.

Furthermore for $\frac{1}{q}+\frac{1}{p}=1$, $||f||_1 \leq (b-a)^{\frac{1}{q}}||f||^p$ and $||f||_p \leq (b-a)^{\frac{1}{p}}||f||_\infty$

The whole finite-approximation of operator – spaces:
$\mathcal{C}_{00} \subseteq \mathcal{C}_0 \subseteq \mathcal{C} \subseteq \mathcal{L}(E;F)$.

Strong / weak norms

A useful tool to determine densities of spaces in each other wrt certain norms is the idea of a strong vs weak norm, since a space being dense in another wrt a strong norm is also dense wrt a weaker norm.

Def a norm is strong compared to weak when $||.||_s \leq c||.||_w$ for some c.

In the below, the constant is omitted.

$$||f||_1 \leq ||f||_p \leq ||f||_\infty$$

$$\|A_{[k]}\|_{\mathcal{L}} \leq \|A_{[k]}\|_{HS} \; the\ Hilbert-Schmidt\ norm\ for\ integral\ operators\ (OP\ theory)$$

Densities

Note that A is dense in Ω if $\bar{A} = \Omega$, with $\bar{A}$ $the\ closure\ of\ A := \{x \in \Omega : \exists (x_n) \to x, x_n \in A\}$

Before we dive in, some useful density theorems:

Approximation theorems

TH. Dense in dense = dense: $A, B \subseteq (\Omega, \mathrm{d})\ with\ A \subseteq \bar{B}\ with\ \bar{A} = \Omega, then\ \bar{B} = \Omega$

TH. Dense = dense in a weaker norm:
TH. Strong vs weak norms: $On\ (\Omega, ||.||_s)\ we\ have\ ||f - f_n||_s \to 0,\ then\ ||f - f_n||_w \to 0, too$
Cor: $A\ dense\ in\ \Omega\ wrt\ ||.||_s \to A\ dense\ in\ \Omega\ wrt\ ||.||_w$

TH. Image of dense is dense in the image: $T: E \to F\ linear, A \subseteq E\ dense, then\ T(A)\ dense\ in\ T(E)$

Density table:

Space 1	Is dense in space 2	Wrt norm (strongest)
c_{00}	l^2	2
c_{00}	c_0	inf
c_{00}	l^p	p
Weierstrass $P[a,b]$	$C[a,b]$	sup
Cor C^∞	C	sup
C_0	C	2
$C_0[a,b]$	$C[a,b]$	p
$C[a,b]$	$L^p(a,b)$	p
C_0^1	C_0	Sup/inf
$C_0^1[a,b]$	$L^p(a,b)$	p
$D := \bigcup C_0^1[a,b], a < b$	$L^p(\mathbb{R})$	p
$PL[a,b]$ piecewise linear	$C[a,b]$	inf
$L_c^p(R)$ compact support	$L^p(R)$	p (<inf)
$C_c^\infty(a,b)$	$C_0[a,b]$	Inf
$C_c^\infty(a,b)$	$L^p(a,b)$	p
$C_c^\infty(R)$	$L^p(R)$	p
$L_c^p(R)$	$L^p(R)$	p
Weierstrass 2.0 $span\{e^{2\pi ins}\}$ trigonometric polynomials	$C_{per}[0,1]$	inf
$C^1[a,b]$	$H^1(a,b)$	H-1
$C_0^1[a,b]$	$H_0^1(a,b)$	H-01
$St([a,b];E)$	$Reg([a,b];E)$	inf

Note: the "wrt norm" column might be abundant, since, given a normed space $(E, \|.\|_E)$, if a space is dense in this larger space, it will always be with respect to the norm $\|.\|_E$. Only when spaces allow for several norms, it is important, but usually it will be obvious.

Operators

Note on spaces such as L^p we naturally pair it with the p-norm.

Operator name	Space to space	Definition	Bounded/operator norm?
Projection	E->E inner product spaces Strictly speaking maps to F a subspace of E	$Pf := \sum < f, e_j > e_j$ With $(e_j) \in E$ an orthonormal system	$\|\|Pf\|\| \leq \|\|f\|\|$
Any operator	$K^d \to F$ From the fields with standard Euclidean norm to any normed space on K^d	…	Yes, CH 2.
Any operator	$F \to E$ with E fin dim	…	Yes, isom-isom K^d->E + above
Shift operators,	On $l^p \to l^p$	$(Lf)(n) = f(n+1)$, $(Rf)(n) = f(n-1)$, Where left deletes the first entry and right adds a 0.	Both with norm 1.
Multiplication operator	Specifically $l^2 \to l^2$	Given $(\lambda_n) \in l^\infty$, $(A_\lambda f) = (\lambda_n f(n))$	Op norm is $\|\|\lambda\|\|_\infty$
Multiplication continuous	$T_m : C \to K$	Given $m \in C$ $T_m f = \int m(s) f(s)$	Op norm is $\|\|m\|\|_1$
	$T_m : C \to (C, \infty)$	$Af = mf$	$\|\|m\|\|_\infty$
Integrator	$J: L^1(a,b) \to (C[a,b], \infty)$	$Jf(t) := \int_a^b \mathbf{1}_{[a,t]} f d\lambda$ $= \int_a^t f(x) dx$	1, $\|\|Jf\|\|_\infty < \|\|f\|\|_1$
Laplace	$\mathcal{L}: L^1(R_+) \to L^\infty(K)$	$(\mathcal{L}f)(t) := \int_{0\to\infty} e^{-ts} f(s)\, ds$	1
Fourier	$\mathcal{F}: L^1(R) \to L^\infty(R)$	$\mathcal{F}f(t) := \int_{-\infty}^{\infty} e^{-its} f(s) ds$	1
Orthogonal projection	$P_F : H \to F$ H Hilbert, F a closed subspace	$*Pf := \sum_j < f, e_j > e_j$ When $\exists (e_j)\ ONS\ in\ H$ $s.t.\ \ F := \overline{span}\{e_j\}$	1. Also: ~, **Parseval:** $\|\|Pf\|\|^2 = \sum_j \|< f, e_j >\|^2$
Derivative	$H^1(a,b) \to L^2(a,b)$	$f \to f'$	Yes
Dirichlet-Laplacian	$\Delta_D : H_0^2(a,b) \to L^2(a,b)$, As well as Δ_D^{-1}.	$\Delta_D u := u''$	Yes

Operator theory

Def Integral operator

$For\ X, Y\ intervals\ on\ \mathbb{R}. An\ operator\ A\ is\ an\ \boldsymbol{integral\ operator}\ if\ \exists function$

$k: X\ x\ Y \to \mathbb{K}$ such that $(Af)(t) = \left(A_{[k]}f\right)(t) := \int_Y k(t,s)f(s)ds.$

Where we call k the **kernel** of the operator.

Furthermore we define the following cross operator:

$$(f \otimes g)(x, y) = f(x)g(y)$$

Where, if f and g are measurable, so is their product. This induces a norm on L(X x Y) as you would expect.

Def $T: E \to F\ is\ invertible\ if\ T\ is\ bijective\ and\ T^{-1} is\ bounded.$

Def Hilbert-Schmidt kernel functions:
For X, Y and k as before, with $k \in L^2(X\ x\ Y)$, i.e., $\int_X \int_Y |k(x,y)|^2 dydx < \infty$,
We call k a Hilbert-Schmidt kernel-function.

Theorem then the induced HS-integral operator $A_{[k]}$ satisfies

$$\left\|A_{[k]}f\right\|_{L_2} \leq \|k\|_{2(X\ x\ Y)} \|f\|_{2(Y)}$$

And, since k is in essence bounded as a HS kernel, we have that the integral operator is bounded.

Moreover: $k\ is\ uniquely\ determined\ by\ A_{[k]}$ (a.e.).

Def HS-norm: $\left\|A_{[k]}\right\|_{HS} := \|k\|_{2(X\ x\ Y)}$

It basically takes the norm of the kernel to define the norm of the corresponding integral operator.

Approximations of operators

From the fact that $F\ Banach \to \mathcal{L}(E; F) is\ Banach$, it follows that $\|ST\| \leq \|S\|\|T\|$.

This allows for **def Strong convergence is when** $(T_n f) \to Tf\ \ \forall f \in F, in\ \|.\|_F.$

Note that $\|T_n f - Tf\| \leq \|T_n - T\|\|f\|$, hence convergence in the operator norm implies strong convergence. So in fact, strong convergence is weaker than convergence in the operator norm.

It is in fact strictly weaker: the projection does converge strongly, $P_n := \sum_{j=1}^n <.,e_j> e_j$ has

$P_n f \to f$ for each $f \in H$. However, the operators never converge in the operator norm.

Defnitions we call $A: E \to F$:

Name	Corresponding space	Definition
Finite rank	C_{00}	$\dim range(A) < \infty$
Finitely approximable	C_0	$\exists (A_n)$ all finite rank: $\|A - A_n\| \to 0$
Compact	C	(f_n) $bnd\ in\ E \to (Af_n)$ has a convergent subsequence.

Examples/theorems:
- HS integral operators are finitely approximable.
- $E, F\ Banach\ \ and\ \ A: E \to F$ is finitely approximable $\to A\ is\ compact$.
- $A \in \mathcal{C}_0 \to AC, DA \in \mathcal{C}_0$ for C,D just linear operators.
- $E, F\ Hilbert \to \mathcal{C}_0(E; F) = \mathcal{C}(E; F)$.

Adjoints

On Hilbert spaces, $A^*: F \to E\ adjoint\ to\ A: E \to F\ bnd\ linear,$

Is such that $< Af, g > = < f, A^*g >$.

Construction of A*: $let\ b\ : H\ x\ K \to \mathbb{K}\ bnd. Then\ \forall f \in H, g \in K, we\ have\ that$

$|b(f,g)| \le c\|f\|\|g\|$, for some c. $Then\ \exists!\ lin. bnd. B: K \to H\ s.t.\ \ b(f,g) = < f, Bg >_H\ \ \forall f, g.$

Even: $\|B\| \le c$. Then we can just take $b(f,g) = < Af, g >$, and by using Riez-Fréchet:

$$A^* = B, c = \|A\|, \qquad A^{**} = A,\ \ and\ \|A\| = \|A^*\|.$$

Theorems:

- A compact -> A* compact.
- A finite rank has A* finite rank.

Lemma: $A\ lin\ bnd, then\ (\ker(A))^\perp = \overline{ran}(A^*)$. So $H = \overline{ran}(A^*) \oplus \ker(A)$.
This is especially nice for self-adjoint operators, which will be useful later.

Theorem: Max-Milgram

If we have:

- $H\ Hilbert, V \subseteq H\ \ a\ linear\ subspace\ s.t.\ (V, <.,.>_V)\ is\ also\ complete,$
- $\exists C \ge 0\ \ s.t.\ \ \|v\|_H \le C\|v\|_V,$
- $\exists a: VxV \to \mathbb{K}\ sesquilinear\ \ s.t.$
    - $a\ is\ bnd, |a(u,v)| \le c\|u\|\|v\|$
    - $a\ is\ coercive, \exists \delta > 0\ \ s.t.\ \ |a(u,v)| \ge \delta\|u\|_V^2,$

Then $\forall f \in H, \exists! u \in V\ \ s.t.\ \ a(u,v) = < f, v >_H,\ \forall v \in V.$

Even, $A: H \to V\ defined\ by\ \ Af \coloneqq u$ has norm $\|A\| \le C/\delta$

Approximate eigenvalues

Def $for\ A \in \mathcal{L}(E),\ \lambda \in \mathbb{C}\ is\ an\ approximate\ eigenvalue\ \tilde{\lambda}\ if\ \exists(f_n) \in E\ \ s.t.$

$\|f_n\| = 1,$ & $\|\lambda f_n - Af_n\| \to 0.$

Here the sequence of functions can be understood as approximate eigenvectors.

Lemma: $Let\ A\ be\ bnd\ on\ E\ Banach.\ If\ (\sigma I - A)\ is\ invertible,\ then\ \sigma\ \ cannot\ be\ an\ \tilde{\lambda}.$
Even, $if\ \ |\sigma| > \|A\|,\ then\ \ \sigma I - A\ \ is\ invertible\ \ (and,\ \sigma\ is\ no\ approx\ eigvalue)$

Theorem:
$Let\ A \in \mathcal{L}(E)\ for\ \ E\ Banach.\ Then\ if\ \lambda! = 0\ is\ an\ approx\ eigenvalue,$
$then\ it\ is\ an\ eigenvalue.$

Furtermore, $\dim\ker(\lambda I - A) < \infty$.

Self-Adjoint: A*=A. THEN:
- $< Af, f > \in \mathbb{R}$
- $\|A\| = |\|A\|| := \sup\{|< Af, f >|\ \ s.t.\ \ \|f\| = 1\}.$ THEN

- All eigenvalues of A are real.
- All eigenvectors are orthogonal.
- $F\ a\ subspace\ \ of\ \ H\ \ s.t.\ A(F) \subseteq F \to\ \ A(F^{\perp}) \subseteq F^{\perp}$, i.e. $\forall f \in F:\ \ Af \in F$
- $\boldsymbol{A = A^*\ compact\ on\ H,\ then\ \exists\lambda \in \mathbb{R}\ s.t.\ \ \|A\| = |\lambda|}.$

Examples: orthogonal projections, multiplication op on l^{∞}, HS integral operators if $\overline{k(x,y)} = k(y,x)$

Theorem: SPECTRAL THEOREM: $For\ \ A = A^* \in \mathcal{L}(H)\ compact.$
Then $\exists(e_n)$ with some indexing set J such that :

- $(e_n)\ is\ an\ \ ONS$
- $\exists(\lambda_n)\ all\ \ in\ \ \mathbb{R}/\{0\}$ with $\lambda_n \to 0$ (in case J = N), s.t.:
- $\boldsymbol{\forall x \in H, Ax = \sum \lambda_n < x, e_n > e_n}$. **Even:** $\boldsymbol{Ae_j = \lambda e_j}$

Uniform Boundedness

$\boldsymbol{Def}: a\ collection\ \ \mathcal{T}\ of\ linear\ op's: E \to F\ \ is\ \boldsymbol{uniformly\ bounded}\ if\ \exists c \geq 0:$
$\|Tf\| \leq c\|f\|,\ \forall f \in E, T \in \mathcal{T}.$

In other words, $\mathcal{T}$ is uniformly bounded if each T is bounded and $\sup\{\|T\|\} < \infty$.
Once can view $\mathcal{T} \subseteq \mathcal{L}(E;F)$ as a bounded subset.

$\boldsymbol{Def}: \mathcal{T}\ is\ pointwise\ bnd\ if\ \|Tf\| \leq \sup_{S \in \mathcal{T}}\{\|S\|\}\,\|f\|.$

Theorem: $E\ Banach, F\ normed \to\ \ \mathcal{T}\ is\ uniformly\ bounded\ iff\ it\ is\ pointwise\ bnd.$

Theorem 15.6 (Banach–Steinhaus[2]). *Let E, F be Banach spaces, and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$ be a sequence such that*

$$Tf := \lim_{n\to\infty} T_n f$$

exists for every $f \in E$. Then T is a bounded operator, $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, and

$$\|T\| \leq \liminf_{n\to\infty} \|T_n\|.$$

Important theorems

General analysis – finite dimensional

Projection and orthonormal system

Lemma 1.10. *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space with associated norm $\|\cdot\|$, and let $e_1, \dots, e_n \in E$ be a finite orthonormal system.*

a) *Let $g = \sum_{j=1}^n \lambda_j e_j$ (with $\lambda_1, \dots, \lambda_n \in \mathbb{K}$) be any linear combination of the e_j. Then*

$$\langle g, e_k \rangle = \sum_{j=1}^n \lambda_j \langle e_j, e_k \rangle = \lambda_k \qquad (k = 1, \dots, n)$$

and $$\|g\|^2 = \sum_{j=1}^n |\lambda_j|^2 = \sum_{j=1}^n |\langle g, e_j \rangle|^2.$$

b) *For $f \in E$ let $Pf := \sum_{j=1}^n \langle f, e_j \rangle e_j$. Then*

$$f - Pf \perp \operatorname{span}\{e_1, \dots, e_n\} \quad \text{and} \quad \|Pf\| \le \|f\|.$$

Bessel's inequality

Combining a) and b) of Lemma 1.10 one obtains **Bessel's inequality**[5]

$$\sum_{j=1}^n |\langle f, e_j \rangle|^2 = \|Pf\|^2 \le \|f\|^2 \qquad (f \in E). \tag{1.1}$$

Also extends to inf dim on Hilbert spaces.

~Projection properties

Exercise 1.8. Let $\{e_1, \dots, e_n\}$ be a finite orthonormal system in an inner product space $(E, \langle\cdot,\cdot\rangle)$, let $F := \operatorname{span}\{e_1, \dots, e_n\}$ and let $P : E \to F$ be the orthogonal projection onto F. Show that the following assertions hold:

a) $PPf = Pf$ for all $f \in E$.

b) If $f, g \in E$ are such that $g \in F$ and $f - g \perp F$, then $g = Pf$.

c) Each $f \in E$ has a *unique* representation as a sum $f = u + v$, where $u \in F$ and $v \in F^\perp$. (In fact, $u = Pf$.)

d) If $f \in E$ is such that $f \perp F^\perp$, then $f \in F$. (Put differently: $(F^\perp)^\perp = F$.)

e) Let $Qf := f - Pf$, $f \in E$. Show that $QQf = Qf$ and $\|Qf\| \le \|f\|$ for all $f \in E$.

Cauchy-Schwarz

Theorem 2.1 (Cauchy–Schwarz Inequality[1,2]). *Let $(E, \langle\cdot,\cdot\rangle)$ be an inner product space with associated norm $\|f\| := \sqrt{\langle f, f \rangle}$ for $f \in E$. Then*

$$|\langle f, g \rangle| \le \|f\| \, \|g\| \qquad (f, g \in E),$$

with equality if and only if f and g are linearly dependent.

For the proof the following is considered:

$$P : E \longrightarrow \operatorname{span}\{g\}, \qquad Pf := \frac{\langle f, g \rangle}{\|g\|^2} g$$

For questions of the form

$\|fg\| \le c\|f\|$, try to write to a form $|\langle f, g \rangle|^2 \le \|f\|^2 \|g\|^2$, i.e. C-S.

Triangle inequalities

$$\|f + g\| \le \|f\| + \|g\|,$$

$$|\|f\| - \|g\|| \le \|f - g\|$$

Lebesque and infinite dimensions

On the Lebesque integral: Dominated convergence

Theorem 7.16 (Dominated Convergence Theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $L^1(X)$ such that $f := \lim_{n\to\infty} f_n$ exists pointwise almost everywhere. If there is $0 \le g \in L^1(X)$ such that $|f_n| \le g$ almost everywhere, for each $n \in \mathbb{N}$, then $f \in L^1(X)$, $\|f_n - f\|_1 \to 0$ and*

$$\int_X f_n \, d\lambda \to \int_X f \, d\lambda.$$

On the inf dimensional projection operator

Theorem 8.8. *Let F be a closed subspace of a Hilbert space H. Then the orthogonal projection P_F has the following properties:*

a) $P_F f \in F$ *and* $f - P_F f \perp F$ *for all* $f \in H$.

b) $P_F f \in F$ *and* $\|f - P_F f\| = d(f, F)$ *for all* $f \in H$.

c) $P_F : H \to H$ *is a bounded linear mapping satisfying* $(P_F)^2 = P_F$ *and*

$$\|P_F f\| \le \|f\| \qquad (f \in H).$$

In particular, either $F = \{0\}$ or $\|P_F\| = 1$.

d) $\operatorname{ran}(P_F) = F$ *and* $\ker(P_F) = F^\perp$.

e) $I - P_F = P_{F^\perp}$, *the orthogonal projection onto* $F^\perp$.

Riesz-Fréchet

Theorem 8.12 (Riesz–Fréchet[1]). *Let H be a Hilbert space and let $\varphi : H \to \mathbb{K}$ be a bounded linear functional on H. Then there exists a unique $g \in H$ such that*

$$\varphi(f) = \langle f, g \rangle \qquad \text{for all } f \in H.$$

Decomposition of L^2

Lemma 10.5. *The space $L^2(a, b)$ decomposes orthogonally into*

$$L^2(a, b) = \mathbb{C}\mathbf{1} \oplus \overline{\{\psi' \mid \psi \in C_0^1[a, b]\}},$$

with $\|\cdot\|_2$-closure on the right-hand side.

Note that C1 is the space of constant functions.

Gives as **corollary the fundamental thm of calc:**

Corollary 10.7. *One has $H^1(a, b) \subseteq C[a, b]$. More precisely, $f \in H^1(a, b)$ if and only if f has a representation*

$$f = Jg + c\mathbf{1}$$

with $g \in L^2(a, b)$ and $c \in \mathbb{K}$. Such a representation is unique, namely

$$g = f' \quad \text{and} \quad c = \frac{\langle f - Jf', \mathbf{1} \rangle}{b - a}.$$

Moreover, the **fundamental theorem of calculus** *holds, i.e.,*

$$\int_c^d f'(s) \, ds = f(d) - f(c) \quad \text{for every interval } [c, d] \subseteq [a, b].$$

Operator theory:

Theorem 11.2. *Let $1 \le p < \infty$. If $f \in L^p(X)$ and $g \in L^p(Y)$, then $f \otimes g \in L^p(X \times Y)$ with $\|f \otimes g\|_{L^p(X\times Y)} = \|f\|_{L^p(X)} \|g\|_{L^p(Y)}$. Moreover, the space*

$$\operatorname{span}\{f \otimes g \mid f \in L^p(X), g \in L^p(Y)\}$$

is dense in $L^p(X \times Y)$.

Fubini

The integral of an integrable function $f \in L^1(X \times Y)$ with respect to two-dimensional Lebesgue measure is computed via iterated integration in either order:

$$\int_{X\times Y} f(\cdot,\cdot)\, d\lambda^2 = \int_X \int_Y f(x,y)\, dy\, dx.$$

This is called **Fubini's theorem**[1] and it includes the statement that if one integrates out just one variable, the function

$$x \longmapsto \int_Y f(x,y)\, dy$$

is again measurable.

Ez integration

Lemma 11.3. *Let $f \in L^1(a,b)$ and $n \in \mathbb{N}$. Then*

$$(J^n f)(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)\, ds \quad \text{for all } t \in [a,b].$$

In particular, J^n is again an integral operator, with kernel function

$$k_n(t,s) = \frac{1}{(n-1)!}\mathbf{1}_{[a,t]}(s)(t-s)^{n-1} \qquad (s,t \in [a,b]).$$

Proof. This is proved by induction and Fubini's theorem. □

Example 11.9 (Integration Operator). The n-th power of the integration operator J on $E = C[a,b]$ is induced by the integral kernel

$$k_n(t,s) = \mathbf{1}_{\{s \le t\}}(t,s)\frac{(t-s)^{n-1}}{(n-1)!}.$$

From this it follows that $\|J^n\|_{\mathcal{L}(E)} = 1/n! \ne 1^n = \|J\|^n$. (See Exercise 11.7.)

The above uses HS-operators from the operator summary.

Lax-Milgram:

Let $a : V \times V \to \mathbb{K}$ be a sesquilinear mapping with the following properties:

1) a is **bounded**, i.e., there is $c > 0$ such that

(12.6) $$|a(u,v)| \le c\,\|u\|_V\, \|v\|_V \qquad (u,v \in V).$$

2) a is **coercive**, i.e., there is $\delta > 0$ such that

$$|a(u,u)| \ge \delta\,\|u\|^2 \qquad (u \in V).$$

(The number δ is called the **coercivity constant**.)

Then we have the following theorem.

Theorem 12.13 (Lax–Milgram[4,5]). *In the situation described above, for each $f \in H$ there is a unique $u \in V$ such that*

$$a(u,v) = \langle f, v\rangle_H \quad \text{for all } v \in V.$$

Moreover, the operator $A : H \to V$ defined by $Af := u$ has norm $\|A\| \le C/\delta$.

Spectral Theorem

Theorem 13.11 (Spectral Theorem). *Let A be a compact self-adjoint operator on a Hilbert space H. Then A is of the form*

(13.1) $$Af = \sum_j \lambda_j \langle f, e_j\rangle e_j \qquad (f \in H)$$

for some (finite or countably infinite) orthonormal system $(e_j)_j$ and real numbers $\lambda_j \ne 0$ satisfying $\lim_{j\to\infty} \lambda_j = 0$. Moreover, $Ae_j = \lambda e_j$ for each j.

More precisely, the orthonormal system is either $(e_j)_{j=1}^N$ for some $N \in \mathbb{N}$ or $(e_j)_{j\in\mathbb{N}}$. Of course, the condition $\lim_{j\to\infty}\lambda_j = 0$ is only meaningful in the second case.

In fact for $A = A^*$ and the spectral thm we get that A is characterized somewhat by a projection:

Let us denote by J the index set for the orthonormal system in the spectral theorem. So $J = \{1,\dots,N\}$ or $J = \mathbb{N}$. Moreover, let

$$P_0 : H \longrightarrow \ker(A)$$

be the orthogonal projection onto the kernel of A and $P_r := I - P_0$ its complementary projection. Then we can write

$$Af = 0 \cdot P_0 f + \sum_{j\in J} \lambda_j \langle f, e_j\rangle e_j$$

for all $f \in H$. This formula is called the **spectral decomposition** of A.

Corollary 13.12. *Let A be as in the spectral theorem (Theorem 13.11). Then the following assertions hold.*

a) $\overline{\operatorname{ran}}(A) = \overline{\operatorname{span}}\{e_j \mid j \in J\}$ *and* $\ker(A) = \{e_j \mid j \in J\}^\perp$.

b) $P_r f = \sum_{j\in J} \langle f, e_j\rangle e_j$ *for all* $f \in H$.

c) *Every nonzero eigenvalue of A occurs in the sequence $(\lambda_j)_{j\in J}$, and its geometric multiplicity is*

$$\dim\ker(\lambda I - A) = \operatorname{card}\{j \in J \mid \lambda = \lambda_j\} < \infty.$$

Baire:

Lemma:

$$Let\ (\Omega, d)\ be\ complete, \quad B_i = B[x_i, r_i],$$
$$B_1 \supseteq B_2 \supseteq B_3 \dots$$

Be a nested sequence of closed balls. $If\ r_n \to 0,$ $then\ x_n \to x\ exists,\ and\ \bigcap_n B_n = \{x\}.$

Theorem 15.1 (Baire). *Let (Ω, d) be a nonempty complete metric space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of Ω such that*

$$\Omega = \bigcup_{n\in\mathbb{N}} A_n.$$

Then there is $n \in \mathbb{N}$ and $x \in \Omega, r > 0$ with $B(x,r) \subseteq A_n$.

Alternatively:

1. $If\ \exists (O_n) \subseteq \Omega\ open\ s.t.\ \overline{O_n} = \Omega\ \forall n \in \mathbb{N}$ $\to \bigcap O_n \ne 0.$
2. $If\ \exists O_n \subseteq \Omega\ open\ s.t.\ \overline{O_n} = \Omega\ \forall n \in \mathbb{N}$ $\to \overline{\bigcap O_n} = \Omega.$

Banach-Steinhaus

Theorem 15.6 (Banach–Steinhaus[2]). *Let E, F be Banach spaces, and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$ be a sequence such that*

$$Tf := \lim_{n\to\infty} T_n f$$

exists for every $f \in E$. Then T is a bounded operator, $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, and

$$\|T\| \le \liminf_{n\to\infty} \|T_n\|.$$

OMP:

Theorem 15.8 (Open Mapping Theorem). *Let E, F be Banach spaces and let $T : E \to F$ be a bounded linear mapping which is surjective. Then there is $a > 0$ such that for each $g \in F$ there is $f \in E$ with $\|f\| \le a\,\|g\|$ and $Tf = g$.*

Alternatively, T maps open subsets of E onto open subsets of F.

$If\ T\ is\ invertible\ then\ the\ inverse\ is\ bounded\ and\ the\ statement\ also\ holds\ for\ T^{-1}.$

Approximate surjectivity:

Theorem 15.11. *Let E, F be Banach spaces, and let $T \in \mathcal{L}(E;F)$. Suppose that there exist $0 \le q < 1$ and $a \ge 0$ such that for every $g \in F$ with $\|g\| \le 1$ there is $f \in E$ such that*

$$\|f\| \le a \quad \text{and} \quad \|Tf - g\| \le q.$$

Then for each $g \in F$ there is $f \in E$ such that $Tf = g$ and $\|f\| \le \frac{a}{1-q}\|g\|$.

So approx. surjectivity implies surjectivity with an estimate of the pre-image.

Closed graph theorem:

Def $T : E \to F$ has a closed graph if

$$\left.\begin{matrix} f_n \to f \\ Tf_n \to g \end{matrix}\right\} \to Tf = g \;\forall (f_n), f \in E, g \in F.$$

In other words, if $graph(T) := \{(f, Tf) \mid f \in E\}$ is closed in the normed VS $E \; x \; F$.

THM: E, F *Banach, then* T *is bnd iff*
$graph(T)$ *is closed*

Tietze: (not very enlightening)

Tietze's Theorem. Let (Ω, d) be a metric space. Any subset $A \subseteq \Omega$ is a metric space with respect to the induced metric, and if $f \in C_b(\Omega)$ is a bounded continuous function, one can consider its restriction

$$Tf := f|_A \in C_b(A)$$

to the set A. The operator $T : C_b(\Omega) \to C_b(A)$ is linear with $\|T\| \le 1$. Tietze's theorem states that if A is closed, then T is surjective.

Theorem 15.15 (Tietze[3]). *Let (Ω, d) a metric space, $A \subseteq \Omega$ a closed subset and $g \in C_b(A; \mathbb{R})$. Then there is $h \in C_b(\Omega; \mathbb{R})$ such that $h|_A = g$ and $\|h\|_\infty = \|g\|_\infty$.*

Uniquely determining a function based on…

Lemma 9.17. *Let $f \in L^1(a, b)$. Then*

$$\|f\|_1 = \sup\left\{ \left| \int_a^b f(s)g(s)\,ds \right| \mid g \in C[a,b],\ \|g\|_\infty \le 1 \right\}. \tag{9.2}$$

In particular, $f = 0$ a.e. if and only if $\int_a^b f(s)g(s)\,ds = 0$ for all $g \in C[a,b]$.

Moments, Fourier coefficients, etc.

TH: In a complete metric space, every *bounded* sequence has a convergent subsequence, *weakly*.

Duality theorems:

Below I conclude with only stuff about duality CH16 since it has to be somewhere, but not in a separate file. Dual def is in the spaces of note.

Does E' always exist (nonzero)?

Often yes: E fin dim, E' same dim
E inner product space, E'=E.

Idea: the dual may be rich enough to distinguish points in E based on evaluation with point in E':
$\forall x \neq y \in E, \exists \varphi \in E' s.t. \; \varphi(x) \neq \varphi(y)$.

Theorem H *Hilbert* $\to H'$ *isometrically isomorphihc to* H. Think of row/col vectors.

Your best mates Riesz-Fréchet say then, as proof:
$h, g \in H$, *any* $\varphi \in H' \sim <., m>$ *for some* $m \in H$.
$\varphi(h) = <h, m> \neq \varphi(g)$ *iff* $<h - g, m> \neq 0$

General case: Hahn-Banach

For $(E, \|.\|), E_0 \subseteq E$ & $\varphi_0 \in E_0'$. Then
$\exists \varphi \in E' \; s.t. \varphi(f) = \varphi_0(f) \; \forall f \in E_0$,

& $\|\varphi\|_{E'} = \|\varphi\|_{E_0'}$.

Cor: every Hilbert space H with countable basis is separable.

Corollaries:

- $\forall f \in E, \exists \; \varphi \in E' s.t. \; \|\varphi\| = 1$ *and* $|\varphi(f)| = \|f\|$.
- $\forall f \in E, \|f\| = \sup_{\|\varphi\|=1} |\varphi(f)|$
- $\forall f \subseteq E$, *have* $\overline{span}(A) = E$ *iff*
  $\forall \varphi \in E'$ *we have* $\varphi|_A = 0 \to \varphi = 0$

If H is a Hilbert space then:

- $\forall f \in H \; \exists \|g\| = 1 \in H : \|f\| = |<f, g>|$
- $\|f\| = \sup_{\|g\|=1} |<f, g>|$ obviously,
- $\overline{span}(A) = H$ *iff* $\forall g \in H$ *have*
  $<f, g> = 0 \; \forall f \in A \to g = 0$.

This last one can be restated as
$A^\perp = \{0\}$ *iff* $g \in H$ *with* $g \in A^\perp \to g = 0$. Trivial.

I cannot be bothered with the pf and the further corollaries.

Sobolev and Poisson

*Note: this document and the one on operators are heavily linked. I want to keep the document on operators and spaces as general as possible. Therefore, this document implicitly draws results/facts from the others.

$$u'' = -f, \;\; u(a) = u(b) = 0$$

To solve this, we need to move to Lebesque spaces and use the **weak derivative:**

Def: weak derivative:

$g \in L^2(a,b)$ is said to be a weak derivative of $f \in L^2(a,b)$ if they satisfy

$\int_a^b g\psi ds = -\int_a^b f\psi' ds$ holds for every test function $\psi \in C_0^1[a,b]$.
This can be rewritten as $< g, \psi > = -< f, \psi' >$

We call the space of all weakly differentiable functions $H^1(a,b)$, the first order Sobolev space.

Variational method for Poisson

We could say that $u \in H^2(a,b)$ since we have a second derivative.

Now rewrite Poisson to $< \psi', u' >_{L^2} = < \psi, f >_{L^2}, \qquad \psi \in C_0^1[a,b]$.

We now constrain u to be in the space H_0^1, which is defined as you would expect, with norm

$$||u||_{H_0^1} := ||u'||_{L^2}$$

Then we rewrite the RHS by using $\varphi: H_0^1(a,b) \to \mathbb{C}, \quad \varphi(v) := < v, f >_{L^2}$

Then Riesz-Fréchet yields a unique $u \in H_0^1(a,b)$ such that

$$< v', u' >_2 =: < v, u >_{H_0^1} = \varphi(v) = < v, f >_2$$

For all $v \in H_0^1(a,b)$. In short, there is a u s.t. $< v', u' >_2 = < v, f >_2$, which holds for all $v \in H_0^1 \supset C_0^1[a,b]$, as required for our problem.

Dirichlet-Laplacian & Hilbert-Schmidt

Def: Dirichlet-Laplacian: $\Delta_D: H_0^2(a,b) \to L^2(a,b), \qquad \Delta_D u := u''$

The importance of writing this as an operator is that there is an inverse operator $\Delta_D^{-1}: L^2 \to H^2$ that turns out to be a HS (kernel) integral operator, which turns out to be bounded, which means that the Poisson problem is well-posed. This is because $-\Delta_D^{-1}$ maps the problem to its unique solution.

Perturbations

$u'' - Tu = -f, \qquad u \in H_0^2(a,b)$. It turns out T 'small enough' is still well-posed.

We use the property that $(\Delta_D\ bijective\ and\ bounden) + (Poisson\ well - posed)$

$\rightarrow\ I - T\Delta_D^{-1}: L^2 \rightarrow L^2\ is\ invertible.$

Note that we can rewrite this problem using the inverse Dirichlet as $-f = (I - T\Delta_D^{-1})\Delta_D u$.

Lemma

Now we can just look at conditions s.t. $(I - A)u = f$ has a unique solution, for $A \in \mathcal{L}(E)$ a perturbation. Without too much work I note that $if\ \ f \in E\ is\ s.t.\ \ u := \sum A^n f\ \ converges\ in\ E,$ $then\ \ u - Au = f$.

Theorem from the above, $\sum\|A^n\| < \infty, \rightarrow\ \ (I - A)\ is\ invertible\ with$

$(I - A)^{-1} = \sum A^n$, the **Neumann series**.

Returning to our problem, **the perturbation is still well-posed if** $\|T\Delta_D^{-1}\| < 1$.

Then in the book there is Volterra which I skip here.

Using compact-self adjoint & Spectral theorem

We can consider the general **eigenvalue equation** $Au - \lambda u = f$

Where $f \in H\ \ Hilbert,\ \lambda \in \mathbb{K},\ A$ is compact self-adjoint. This is solvable under the following theorem (with (e_j) from the spectral theorem):

Theorem 13.13 (Fredholm Alternative[1]). *In the situation above, precisely one of the following cases holds:*

1) *If* $\lambda \neq 0$ *is different from every* λ_j, *then* $(\lambda \mathrm{I} - A)$ *is invertible and*

$$u := (A - \lambda \mathrm{I})^{-1} f = -\frac{1}{\lambda} P_0 f + \sum_{j\in J} \frac{1}{\lambda_j - \lambda} \langle f, e_j \rangle e_j$$

is the unique solution to (13.2).

2) *If* $\lambda \neq 0$ *is an eigenvalue of* A, *then* (13.2) *has a solution if and only if* $f \perp \ker(\lambda \mathrm{I} - A)$. *In this case a particular solution is*

$$u := -\frac{1}{\lambda} P_0 f + \sum_{j\in J_\lambda} \frac{1}{\lambda_j - \lambda} \langle f, e_j \rangle e_j,$$

where $J_\lambda := \{j \in J \mid \lambda_j \neq \lambda\}$.

3) *If* $\lambda = 0$, *then* (13.2) *is solvable if and only if* $f \in \mathrm{ran}(A)$; *in this case one particular solution is*

$$u := \sum_{j\in J} \frac{1}{\lambda_j} \langle f, e_j \rangle e_j,$$

this series being indeed convergent.

Let us consider then $\Delta_D u = -f$, with its solution $-\Delta_D^{-1} f = Af = \int g(.,s) f(s) ds$, $g(s)$ the Green function. Note the following:

- $A\ is\ a\ HS - operator\ and\ hence\ compact$
- $k\ is\ symmetric\ and\ real - valued, hence\ A\ is\ self\ adjoint$
- $\ker(A) = \{0\}$ by construction.

Hence: we can apply the spectral theorem if we can find the eigenvalues and eigenvectors of A.

So first, to determine the eigenvalues/eigenvectors:

Lemma 14.1. *Let* $\lambda \neq 0$, $\mu = -1/\lambda$. *Then*

$$f \in L^2(a,b) \quad and \quad Af = \lambda f \quad \Longleftrightarrow \quad f \in \mathrm{dom}(\Delta_D) \quad and \quad \Delta_D f = \mu f.$$

Moreover, in this case either $f = 0$ *or* $\lambda > 0$.

Where $dom(\Delta_D) = H_0^2(a,b)$.

By ez DE-theory we have $Au = \lambda u, \lambda > 0 \; iff$

$u = \alpha \cos\left(\frac{t}{\sqrt{\lambda}}\right) + \beta \sin(t/\sqrt{\lambda})$. Now this solution can be further sharpened by the boundary conditions. In particular, letting $a = 0, b = 1, u(0) = 0 \rightarrow \alpha = 0$.
Then $u(1) = 0, \beta \neq 0 \rightarrow \sin\left(\frac{t}{\sqrt{\lambda}}\right) = 0 \quad \rightarrow \quad \lambda_n = \frac{1}{n^2\lambda^2}, e_n = \frac{1}{\sqrt{2}}\sin(n\pi t)$ (normalized).

Furthermore, A is injective to L^2, so the system (e_n) is an orthonormal basis for $L^2(0,1)$, and:

$$(Af)(t) = \int_0^1 g(t,s)f(s)\,ds = \sum_{n=1}^{\infty}\left(\frac{1}{2n^2\pi^2}\int_0^1 f(s)\sin(n\pi s)\,ds\right)\sin(n\pi t)$$

Which converges by the theory in L^2 but not necessarily pointwise. However it can be shown that it does in fact converge uniformly in t=[0,1]. Furthermore:

$$g(t,s) = \sum_{n=1}^{\infty}\frac{\sin(n\pi \cdot t)\sin(n\pi \cdot s)}{2n^2\pi^2}$$

As an absolutely convergent series in $C([0,1]x[0,1])$.

Schrödinger operator & Strum Liouville equation

Is just a perturbation of the Dirichlet-Laplacian with a multiplication operator:

$Lu = -u'' + qu \;\; for\; some\; q \in C[0,1]$ a positive continuous function, called the potential. Once again the domain is $dom(L) = H_0^2(0,1)$. We can consider the eigenvalues of L.

$Lu = \lambda u, \; then \;\; u \in C^2[0,1] \; and \; either \;\; u = 0 \;\; or \; \lambda < 0$. In particular, $L \; is \; injective$ (1-1).

Sturm-Liouville:

$Lu = f$, is well-posed for $f \in L^2(0,1)$, i.e., $L: H_0^1 \rightarrow L^2$ is bijective with bounded inverse. To this end, we define the new inner product $a(u,v) := < u', v' >_2 + < qu, v >_2$. The induced norm is equivalent to the usual norm on $H_0^1, \;\; and \; (H_0^1, \|.\|_a)$ is a Hilbert space. Then the mapping

$v \mapsto < v, f >_2$ is bounded, and by Riesz-Fréchet $\exists! \, u \in H_0^1 \;\; s.t. \;\; a(u,v) = < f, v > \forall v \in H_0^1$.

For $v \in C_0^1, then, \;\; u \in H_0^2, \;\; and \;\; Lu = f, \;\; and \; L \; is \; bijective$. It can also be shown that L^{-1} is bnd. In the book they show L^{-1} can be found (as a HS-integral operator) but that is cumbersome and skipped.

Fourier analysis

From Chapter 1:

The number

$$\widehat{f}(n) := \langle f, e_n \rangle = \int_0^1 f(t)\overline{e_n(t)}\,\mathrm{d}t = \int_0^1 f(t)\mathrm{e}^{-2\pi i n t}\,\mathrm{d}t$$

is called the n-th **Fourier coefficient** of f. Note that n ranges over the whole set of integers $\mathbb{Z}$. Bessel's inequality in this context reads

$$\sum_{n=-N}^{N} |\widehat{f}(n)|^2 \le \|f\|_2^2 = \int_0^1 |f(t)|^2\,\mathrm{d}t. \tag{1.3}$$

With $e_n(t) = e^{2n\pi i t}$

This can be extended after Lebesque and Hilbert to inf dim:

For a function $f \in \mathrm{L}^1(\mathbb{R})$ its **Fourier transform** $\mathcal{F}f$ is defined by

$$(\mathcal{F}f)(t) = \int_{\mathbb{R}} f(s)\,\mathrm{e}^{-\mathrm{i}st}\,\mathrm{d}s \qquad (t \in \mathbb{R}). \tag{9.3}$$

The integral is well-defined since

$$\int_{\mathbb{R}} \left|f(s)\,\mathrm{e}^{-\mathrm{i}st}\right|\,\mathrm{d}s = \int_{\mathbb{R}} |f(s)|\,\mathrm{d}s = \|f\|_1 < \infty.$$

Moreover, by the triangle inequality for integrals it follows that $|(\mathcal{F}f)(t)| \le \|f\|_1$ and taking the supremum over $t \in \mathbb{R}$ we arrive at

$$\|\mathcal{F}f\|_\infty \le \|f\|_1 \qquad (f \in \mathrm{L}^1(\mathbb{R})). \tag{9.4}$$

This shows that the Fourier transform is a bounded linear operator

$$\mathcal{F} : (\mathrm{L}^1(\mathbb{R}), \|\cdot\|_1) \longrightarrow (\mathcal{B}(\mathbb{R}), \|\cdot\|_\infty).$$

Applying the dominated convergence theorem one can show that $\mathcal{F}f$ is a continuous function for every $f \in \mathrm{L}^1(\mathbb{R})$; see Exercise 7.21. Regarding the asympotic behaviour of $\mathcal{F}f(t)$ for large values of $|t|$ we have the following analogue of Theorem 9.19.

Theorem 9.20 (Riemann–Lebesgue[5]). *If $f \in \mathrm{L}^1(\mathbb{R})$, then $\mathcal{F}f \in \mathrm{C}(\mathbb{R})$ and*

$$\lim_{|t|\to\infty} (\mathcal{F}f)(t) = 0.$$

Theorem 15.7 (Du Bois-Reymond). *There exists a function $f \in \mathrm{C}_{\mathrm{per}}[0,1]$ such that its partial Fourier series at $t = 0$,*

$$S_n f(0) = \sum_{k=-n}^{n} \widehat{f}(k) e^{2\pi i k t}\Big|_{t=0} = \sum_{k=-n}^{n} \widehat{f}(k) \qquad (n \in \mathbb{N})$$

does not converge to $f(0)$.

Need the Dirichlet kernel:

Define the **Dirichlet kernel**

$$D_n(s) := \frac{\sin(2n+1)\pi s}{\sin \pi s},$$

so that $T_n f = \int_0^1 D_n(s) f(s)\,\mathrm{d}s$ for $f \in E$. We claim that

$$\|T_n\| = \int_0^1 |D_n(s)|\,\mathrm{d}s.$$

Proof. We consider the linear functionals

$$T_n : \mathrm{C}_{\mathrm{per}}[0,1] \longrightarrow \mathbb{C}, \qquad T_n f := (S_n f)(0)$$

for $n \in \mathbb{N}$. Then

$$T_n f = \sum_{k=-n}^{n} \int_0^1 \mathrm{e}^{2\pi i k s} f(s)\,\mathrm{d}s = \int_0^1 \frac{\sin(2n+1)\pi s}{\sin \pi s} f(s)\,\mathrm{d}s$$

Then some stuff and some more stuff with which I can't be bothered and then $\|T_n\|$ is the harmonic series which diverges.

For more on Fourier, see the appendix.