AFA – Applied Functional Analysis

Know-by-heart

Questions

This document and its solutions are intended as follows: for the AFA oral exam, it is often the case that you are asked to recite theorems and their proofs by heart. That is why the reader should take theorems from this reader, if these are named, try to write them out by heart, and them try to reproduce the proof from memory. Actual proofs and definitions are provided in the solution document.

The idea is that if you know all of this by heart, you'll do well on the oral exam. Good luck.

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Lemma 1.11 (Gram[6]–Schmidt[7]). *Let $N \in \mathbb{N} \cup \{\infty\}$ and let $(f_n)_{1 \le n < N}$ be a linearly independent set of vectors in an inner product space E. Then there is an orthonormal system $(e_n)_{1 \le n < N}$ in E such that*

$$\operatorname{span}\{e_j \mid 0 \le j < n\} = \operatorname{span}\{f_j \mid 0 \le j < n\} \qquad \text{for all } n \le N.$$

Theorem 2.1 (Cauchy–Schwarz Inequality[1,2]). *Let $(E, \langle \cdot, \cdot \rangle)$ be an inner product space with associated norm $\|f\| := \sqrt{\langle f, f \rangle}$ for $f \in E$. Then*

$$|\langle f, g \rangle| \le \|f\| \, \|g\| \qquad (f, g \in E),$$

with equality if and only if f and g are linearly dependent.

Example of bounded linear mapping?

Example 3.12 (Scale of ℓ^p-Spaces). Let $f : \mathbb{N} \to \mathbb{K}$ be any scalar sequence. We claim that

$$\|f\|_\infty \le \|f\|_2 \le \|f\|_1 \qquad \text{in } [0, \infty]. \tag{3.5}$$

Example 3.14 (p-Norms on $\mathrm{C}[a, b]$). *For each interval $[a, b] \subseteq \mathbb{R}$ we have*

$$\|f\|_1 \le \sqrt{b - a}\, \|f\|_2 \quad \text{and} \quad \|f\|_2 \le \sqrt{b - a}\, \|f\|_\infty \tag{3.6}$$

for all $f \in \mathrm{C}[a, b]$.

Theorem 3.22 (Weierstrass). *Let $[a,b]$ be a compact interval in $\mathbb{R}$. Then the space of polynomials $\mathrm{P}[a,b]$ is dense in $\mathrm{C}[a,b]$ with respect to the supremum norm.*

We let, for $k \in \mathbb{N}$ or $k = \infty$,

$$\mathrm{C}^k[a,b] := \{f : [a,b] \longrightarrow \mathbb{K} \mid f \text{ is } k\text{-times continuously differentiable}\}.$$

Since polynomials are infinitely differentiable, Weierstrass' theorem implies that $\mathrm{C}^\infty[a,b]$ is dense in $\mathrm{C}[a,b]$.

Only know by heart

Exercise 3.4. Show that the inclusions

$$\ell^1 \subseteq \ell^2 \subseteq \ell^\infty,$$

are all strict. Give an example of sequences $(f_n)_{n\in\mathbb{N}}, (g_n)_{n\in\mathbb{N}}$ in ℓ^1 with

$$\|f_n\|_\infty \to 0,\ \|f_n\|_2 \to \infty \quad \text{and} \quad \|g_n\|_2 \to 0,\ \|g_n\|_1 \to \infty.$$

Lemma 4.9 (Second triangle inequality). *Let (Ω, d) be a metric space. Then*

$$|d(x,z) - d(y,w)| \le d(x,y) + d(z,w) \qquad \text{for all } x,y,z,w \in \Omega.$$

Theorem 4.16. *A linear mapping $T : E \to F$ between two normed spaces $(E, \|\cdot\|_E)$ and $(F, \|\cdot\|_F)$ is continuous if and only if it is bounded.*

Lemma 4.20. *Let $A \subseteq \Omega$ be subset of a metric space (Ω, d). If A is compact, then A is closed in Ω; and if Ω is compact and A is closed in Ω, then A is compact.*

Theorem 4.21 (Bolzano[1]–Weierstrass). *With respect to the Euclidean metric on $\mathbb{K}^d$ a subset $A \subseteq \mathbb{K}^d$ is (sequentially) compact if and only if it is closed and bounded.*

Only know by heart.

Theorem 4.29. *Let E be a finite dimensional linear space. Then all norms on E are equivalent.*

Theorem 4.37. *A normed space E is separable if and only if there is a countable set $M \subseteq E$ such that* $\operatorname{span}(M)$ *is dense in E.*

Only know by heart.

Corollary 4.34. *In each infinite-dimensional normed space E there is a sequence of unit vectors $(f_n)_{n\in\mathbb{N}}$ such that $\|f_n - f_m\| \geq 1$ for all $n, m \in \mathbb{N}$ with $n \neq m$.*

Only know by heart.

Example 5.10. *Every finite-dimensional normed space is a Banach space.*

Example 5.11. *Let Ω be a nonempty set. Then $(\mathcal{B}(\Omega), \|\cdot\|_\infty)$ is a Banach space.*

Note here $\mathcal{B}$ means the set of all bounded functions, not a ball.

Example 5.13. *The space* $\mathrm{C}[a, b]$ *is a Banach space with respect to the supremum norm $\|\cdot\|_\infty$.*

Definition 7.1. The **Lebesgue outer measure** of a set $A \subseteq \mathbb{R}$ is

$$\lambda^*(A) := \inf \sum_{n=1}^{\infty} |Q_n|$$

where the infimum is taken over all sequences of intervals $(Q_n)_{n\in\mathbb{N}}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} Q_n$. (Such a sequence is called a **cover** of A.)

Know by heart

Theorem 7.16 (Dominated Convergence Theorem). *Let $(f_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{L}^1(X)$ such that $f := \lim_{n\to\infty} f_n$ exists pointwise almost everywhere. If there is $0 \leq g \in \mathrm{L}^1(X)$ such that $|f_n| \leq g$ almost everywhere, for each $n \in \mathbb{N}$, then $f \in \mathrm{L}^1(X)$, $\|f_n - f\|_1 \to 0$ and*

$$\int_X f_n \,\mathrm{d}\lambda \to \int_X f \,\mathrm{d}\lambda.$$

Theorem 7.18 (Completeness of L^1). *The space $\mathrm{L}^1(X)$ is a Banach space. More precisely, let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\mathrm{L}^1(X)$. Then there are functions $f, g \in \mathrm{L}^1(X)$ and a subsequence $(f_{n_k})_{k\in\mathbb{N}}$ such that*

$$|f_{n_k}| \leq g \quad \text{a.e.} \quad \text{and} \quad f_{n_k} \to f \quad \text{a.e.}.$$

Furthermore, $\|f_n - f\|_1 \to 0$.

Theorem 7.22 (Hölder's Inequality). *Let q be the dual exponent defined by $1/p + 1/q = 1$. If $f \in \mathrm{L}^p(X)$ and $g \in \mathrm{L}^q(X)$, then $fg \in \mathrm{L}^1(X)$ and*

$$\left| \int_X fg \,\mathrm{d}\lambda \right| \leq \|f\|_p \, \|g\|_q .$$

Density. Finally, we return to our starting point, namely the question of a natural "completion" of $\mathrm{C}[a,b]$ with respect to $\|\cdot\|_1$ or $\|\cdot\|_2$. If $X=[a,b]$ is a finite interval, $1<p<\infty$ and $1/p+1/q=1$, then one has

$$\mathrm{C}[a,b] \subseteq \mathrm{C}_b(a,b) \subseteq \mathrm{L}^\infty(a,b) \subseteq \mathrm{L}^p(a,b) \subseteq \mathrm{L}^1(a,b)$$

with

$$\begin{aligned} \|f\|_1 &\le (b-a)^{1/q}\,\|f\|_p && \text{for all } f\in \mathrm{L}^p(a,b),\\ \|f\|_p &\le (b-a)^{1/p}\,\|f\|_{\mathrm{L}^\infty} && \text{for all } f\in \mathrm{L}^\infty(a,b),\\ \|f\|_\infty &= \qquad \|f\|_{\mathrm{L}^\infty} && \text{for all } f\in \mathrm{C}_b(a,b). \end{aligned}$$

(The proof is an exercise.) The following result gives the desired answer to our question, but once again, we can do nothing but quote the result without being able to provide a proof here. Ex.7.15

Theorem 7.24. *The space* $\mathrm{C}[a,b]$ *is* $\|\cdot\|_p$*-dense in* $\mathrm{L}^p(a,b)$ *for* $1\le p<\infty$.

Note: The space $\mathrm{C}_b(a,b)$ is *not* $\|\cdot\|_{\mathrm{L}^\infty}$-dense in $\mathrm{L}^\infty(a,b)$. Ex.7.16

Theorem 8.5. *Let H be an inner product space, and let $A\neq\emptyset$ be a complete convex subset of H. Furthermore, let $f\in H$. Then there is a unique vector $P_A f := g\in A$ with $\|f-g\| = d(f,A)$.*

Hint: parallelogram identity

Corollary 8.10 (Orthogonal Decomposition). *Let H be a Hilbert space, and let $F\subseteq H$ be a closed linear subspace. Then every vector $f\in H$ can be written in a unique way as $f=u+v$ where $u\in F$ and $v\in F^\perp$.*

Theorem 8.12 (Riesz–Fréchet[1]). *Let H be a Hilbert space and let $\varphi : H\to \mathbb{K}$ be a bounded linear functional on H. Then there exists a unique $g\in H$ such that*

$$\varphi(f) = \langle f,g\rangle \qquad \textit{for all } f\in H.$$

Theorem 8.13. *Let $(H, \langle\cdot,\cdot\rangle)$ be an inner product space and let $(f_n)_{n\in\mathbb{N}}$ be a sequence of pairwise orthogonal elements of H. Consider the statements*

(i) *The series $f := \sum_{n=1}^{\infty} f_n$ converges in H.*

(ii) $\sum_{n=1}^{\infty} \|f_n\|^2 < \infty$.

Then (i) *implies* (ii) *and one has* **Parseval's identity**[2]

$$\|f\|^2 = \sum_{n=1}^{\infty} \|f_n\|^2 . \tag{8.1}$$

If H is a Hilbert space, then (ii) *implies* (i).

Only proof (i), optional (ii)

Theorem 8.15. *Let H be a Hilbert space, let $(e_j)_{j\in\mathbb{N}}$ be an orthonormal system in H, and let $f \in H$. Then one has* **Bessel's inequality**

$$\sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2 \le \|f\|^2 < \infty. \tag{8.2}$$

Moreover, the series

$$Pf := \sum_{j=1}^{\infty} \langle f, e_j\rangle\, e_j$$

is convergent in H, and $Pf = P_F f$ is the orthogonal projection of f onto the closed subspace

$$F := \overline{\operatorname{span}}\{e_j \mid j \in \mathbb{N}\}.$$

Finally, one has **Parseval's identity** $\quad \|Pf\|^2 = \sum_{j=1}^{\infty} |\langle f, e_j\rangle|^2$.

Lemma 9.17. *Let $f \in \mathrm{L}^1(a, b)$. Then*

$$\|f\|_1 = \sup\left\{ \left| \int_a^b f(s)g(s)\,\mathrm{d}s \right| \mid g \in \mathrm{C}[a,b],\ \|g\|_\infty \le 1 \right\}. \tag{9.2}$$

In particular, $f = 0$ a.e. if and only if $\int_a^b f(s)g(s)\,\mathrm{d}s = 0$ for all $g \in \mathrm{C}[a,b]$.

Only know by heart

Lemma 10.5. *The space* $\mathrm{L}^2(a,b)$ *decomposes orthogonally into*

$$\mathrm{L}^2(a,b) = \mathbb{C}\mathbf{1} \oplus \overline{\{\psi' \mid \psi \in \mathrm{C}_0^1[a,b]\}},$$

with $\|\cdot\|_2$*-closure on the right-hand side.*

Corollary 10.7. *One has* $\mathrm{H}^1(a,b) \subseteq \mathrm{C}[a,b]$*. More precisely,* $f \in \mathrm{H}^1(a,b)$ *if and only if* f *has a representation*

$$f = Jg + c\mathbf{1}$$

with $g \in \mathrm{L}^2(a,b)$ *and* $c \in \mathbb{K}$*. Such a representation is unique, namely*

$$g = f' \quad \text{and} \quad c = \frac{\langle f - Jf', \mathbf{1}\rangle}{b-a}.$$

Moreover, the **fundamental theorem of calculus** *holds, i.e.,*

$$\int_c^d f'(s)\,\mathrm{d}s = f(d) - f(c) \quad \text{for every interval } [c,d] \subseteq [a,b].$$

Only know by heart: this is how L^2 relates to derivatives, and how H^1 relates to L^2.

Lemma 10.10 (Poincaré Inequality[4])**.** *There is a constant* $C \geq 0$ *depending on* $b-a$ *such that*

$$\|u\|_{\mathrm{L}^2} \leq C\|u'\|_{\mathrm{L}^2} \tag{10.8}$$

for all $u \in \mathrm{H}_0^1(a,b)$*. In particular,* (10.7) *is an inner product and* $\|\cdot\|_{\mathrm{H}_0^1}$ *is a norm on* $\mathrm{H}_0^1(a,b)$*.*

Lemma 11.3. *Let $f \in L^1(a,b)$ and $n \in \mathbb{N}$. Then*

$$(J^n f)(t) = \frac{1}{(n-1)!}\int_a^t (t-s)^{n-1} f(s)\,\mathrm{d}s \quad \text{for all } t \in [a,b].$$

In particular, J^n is again an integral operator, with kernel function

$$k_n(t,s) = \frac{1}{(n-1)!}\mathbf{1}_{[a,t]}(s)(t-s)^{n-1} \qquad (s,t \in [a,b]).$$

Proof. This is proved by induction and Fubini's theorem. □

Example 11.9 (Integration Operator). The n-th power of the integration operator J on $E = \mathrm{C}[a,b]$ is induced by the integral kernel

$$k_n(t,s) = \mathbf{1}_{\{s \le t\}}(t,s)\frac{(t-s)^{n-1}}{(n-1)!}.$$

From this it follows that $\|J^n\|_{\mathcal{L}(E)} = {}^1\!/_{n!} \neq 1^n = \|J\|^n$. (See Exercise 11.7.)

Only know by heart; Fubini only tells you how to integrate a function of 2 variables.

Definitions 12.1:

- **Finite dimensional operator?**
- **Finitely approximable operators?**
- **Compact operators?**

Corollary 12.10. *Let H, K be Hilbert spaces, and let $A : H \to K$ be a bounded linear operator. Then there is a unique bounded linear operator $A^* : K \to H$ such that*

$$\langle Af, g\rangle_K = \langle f, A^*g\rangle_H \quad \text{for all} \quad f \in H,\ g \in K.$$

Furthermore, one has $(A^*)^* = A$ and $\|A^*\| = \|A\|$.

Def 13.2: Approximate eigenvalues

Lemma 13.4. *Let A be a bounded operator on the Banach space E. If $\lambda \mathrm{I} - A$ is invertible, then λ cannot be an approximate eigenvalue. If $|\lambda| > \|A\|$, then $\lambda \mathrm{I} - A$ is invertible.*

Theorem 13.8. *Let A be a bounded self-adjoint operator on a Hilbert space A. Then $\langle Af, f\rangle \in \mathbb{R}$ for all $f \in H$ and*

$$\|A\| = \|\!| A |\!\| := \sup\{|\langle Af, f\rangle| \mid f \in H, \|f\| = 1\}.$$

Only prove $|||A||| \leq \|A\|$

Theorem 13.11 (Spectral Theorem). *Let A be a compact self-adjoint operator on a Hilbert space H. Then A is of the form*

$$Af = \sum_j \lambda_j \langle f, e_j\rangle e_j \qquad (f \in H) \tag{13.1}$$

for some (finite or countably infinite) orthonormal system $(e_j)_j$ and real numbers $\lambda_j \neq 0$ satisfying $\lim_{j\to\infty} \lambda_j = 0$. Moreover, $Ae_j = \lambda e_j$ for each j.

More precisely, the orthonormal system is either $(e_j)_{j=1}^N$ for some $N \in \mathbb{N}$ or $(e_j)_{j\in\mathbb{N}}$. Of course, the condition $\lim_{j\to\infty} \lambda_j = 0$ is only meaningful in the second case.

Lemma 13.10. *Let A be a compact self-adjoint operator on a Hilbert space. Then A has an eigenvalue λ such that $|\lambda| = \|A\|$.*

Only know by heart for the Spectral Theorem

Lemma 13.9. *Let A be a self-adjoint operator on a Hilbert space. Then the following assertions hold.*

a) *Every eigenvalue of A is real.*

b) *Eigenvectors with respect to different eigenvalues are orthogonal to each other.*

c) *If F is an A-invariant subspace of H, then $F^{\perp}$ is also A-invariant.*

May be used for the Spectral Theorem; A-invariant implies $A(F) \subseteq F$

Theorem 13.11 (Spectral Theorem). *Let A be a compact self-adjoint operator on a Hilbert space H. Then A is of the form*

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for some (finite or countably infinite) orthonormal system $(e_j)_j$ and real numbers $\lambda_j \neq 0$ satisfying $\lim_{j\to\infty} \lambda_j = 0$. Moreover, $Ae_j = \lambda e_j$ for each j.

More precisely, the orthonormal system is either $(e_j)_{j=1}^N$ for some $N \in \mathbb{N}$ or $(e_j)_{j\in\mathbb{N}}$. Of course, the condition $\lim_{j\to\infty} \lambda_j = 0$ is only meaningful in the second case.

Theorem 15.1 (Baire). *Let (Ω, d) be a nonempty complete metric space and let $(A_n)_{n\in\mathbb{N}}$ be a sequence of closed subsets of Ω such that*

$$\Omega = \bigcup_{n\in\mathbb{N}} A_n.$$

Then there is $n \in \mathbb{N}$ and $x \in \Omega, r > 0$ with $\mathrm{B}(x,r) \subseteq A_n$.

May use the following:

Lemma 15.2 (Principle of Nested Balls). *Let (Ω, d) be a complete metric space, and let*

$$\mathrm{B}[x_1, r_1] \supseteq \mathrm{B}[x_2, r_2] \supseteq \mathrm{B}[x_3, r_3] \supseteq \dots$$

be a nested *sequence of closed balls in it. If $r_n \to 0$, then $x := \lim_{n\to\infty} x_n$ exists and*

$$\bigcap_{n\in\mathbb{N}} \mathrm{B}[x_n, r_n] = \{x\}. \tag{15.1}$$

Definition 15.3

- **Uniform boundedness of a collection $\mathcal{T}$ of linear mappings E to F**
- **Pointwise boundedness of the same family**

Theorem 15.4 (Uniform Boundedness Principle). *Let E be a Banach space, let F be a normed space, and let $\mathcal{T}$ be a collection of bounded linear operators from E to F. Then $\mathcal{T}$ is uniformly bounded if and only if it is pointwise bounded.*

Theorem 15.6 (Banach–Steinhaus[2]). *Let E, F be Banach spaces, and let $(T_n)_{n\in\mathbb{N}} \subseteq \mathcal{L}(E;F)$ be a sequence such that*

$$Tf := \lim_{n\to\infty} T_n f$$

exists for every $f \in E$. Then T is a bounded operator, $(T_n)_{n\in\mathbb{N}}$ is uniformly bounded, and

$$\|T\| \le \liminf_{n\to\infty} \|T_n\| .$$

Proof. For each $f \in E$, since $(T_n f)_{n\in\mathbb{N}}$ converges, also $(\|T_n f\|)_{n\in\mathbb{N}}$ converges, and therefore $\sup_{n\in\mathbb{N}} \|T_n f\| < \infty$. By the uniform boundedness principle, $\sup_{n\in\mathbb{N}} \|T_n\| < \infty$. If $f \in E$ with $\|f\| \le 1$, then by the continuity of the norm,

$$\|Tf\| = \lim_{n\to\infty} \|T_n f\| = \liminf_{n\to\infty} \|T_n f\| \le \liminf_{n\to\infty} \|T_n\| .$$

Taking the supremum over all such f concludes the proof. □

Only really necessary to know the theorem

Theorem 15.8 (Open Mapping Theorem). *Let E, F be Banach spaces and let $T : E \to F$ be a bounded linear mapping which is surjective. Then there is $a > 0$ such that for each $g \in F$ there is $f \in E$ with $\|f\| \le a\,\|g\|$ and $Tf = g$.*

May use in-between theorem "Rough surjective + approx. pre-images" = surjective

Theorem 15.11. *Let E, F be Banach spaces, and let $T \in \mathcal{L}(E; F)$. Suppose that there exist $0 \le q < 1$ and $a \ge 0$ such that for every $g \in F$ with $\|g\| \le 1$ there is $f \in E$ such that*

$$\|f\| \le a \quad \text{and} \quad \|Tf - g\| \le q.$$

Then for each $g \in F$ there is $f \in E$ such that $Tf = g$ and $\|f\| \le \frac{a}{1-q}\,\|g\|$.

Definition of $graph(T)$

Theorem 15.12 (Closed Graph Theorem). *If E, F are Banach spaces and $T : E \to F$ is a linear mapping, then T is bounded if and only if it has a closed graph.*

Definition: separable normed space

Theorem 16.2 (Hahn–Banach,[1] Separable Case). *Let E be a separable normed space over the scalar field $\mathbb{K}$. Let $E_0 \subseteq E$ be a subspace and $\varphi_0 \in E_0'$ a bounded linear functional on E_0. Then there is an extension $\varphi \in E'$ of φ_0 to all of E with $\|\varphi\| = \|\varphi_0\|$.*

May use

Lemma 16.1. *Let E be a real linear space and let $p : E \to \mathbb{R}$ be a sublinear functional. Furthermore, let $F \subseteq E$ be a linear subspace, $\varphi : F \to \mathbb{R}$ a linear mapping with $\varphi \le p$ on F. Given any $h \in E \setminus F$ there is $\alpha \in \mathbb{R}$ such that the definition*

$$F_1 := F \oplus \mathbb{R}\,h, \quad \varphi_1(f + th) := \varphi(f) + \alpha\,t \qquad (t \in \mathbb{R},\ f \in F)$$

yields a linear mapping $\varphi_1 : F_1 \to \mathbb{R}$ with $\varphi_1|_F = \varphi$ and $\varphi_1 \le p$ on F_1.

Examples and counter-examples:

- Compact spaces
- Complete spaces
- Finite rank operators
- Finitely approximable operators
- Compact operators
- Adjoint operators
- Operators which have eigenvalues
- Operators with approximate eigenvalues
- Self-adjoint operators
- An application of the spectral theorem for self-adjoint operators
- Separable spaces