Applied Functional Analysis (191506302)

Tuesday, January 31, 2012, 08.45-11.45

- An explanation to every answer is required
- You can make use of a calculator
- (Part of) the score to exercises 3(b), 3(c), and 5(d) may be earned by homeworks; see the score table at the end

1. The linear space $C[0,1]$ of continuous real functions on the interval $[0,1]$ is endowed with

$$\|f\|_\infty = \max_{0\le t\le 1} |f(t)|$$

and with

$$\|f\| = |f(0)| + \int_0^1 |f(t)|dt$$

   (a) Show that $\|f\|$ defines a norm on $C[0,1]$.

   (b) Prove that $\| \ \|_\infty$ and $\| \ \|$, are non-equivalent norms on $C[0,1]$.

   (c) Give an example of a linear functional $A : C[0,1] \longrightarrow \mathbb{R}$ which is continuous with respect to $\| \ \|_\infty$, but not with respect to $\| \ \|$.

   (d) From the course it is known that $C[0,1]$ endowed with $\| \ \|_\infty$ is a Banach space. Is $C[0,1]$ also complete with respect to $\| \ \|$?

2. By $L^2(0,2\pi)$ we denote the real vector space of all (classes of) real square-integrable functions on $(0,2\pi)$, endowed with the usual inner product

$$(f,g) = \int_{x=0}^{2\pi} f(x)g(x)dx$$

   Let D be the closed linear subspace, $D = \{f \in L^2(0,2\pi) | \int_0^{2\pi} f(x)dx = 0\}$.

   (a) Determine $D^\perp$ (the 'orthoplement' of D in $L^2(0,2\pi)$).

   (b) Find the best approximation in D to $g(x) = x^2$.

   (c) Find $f \in L^2(0,2\pi)$ with minimal norm satisfying both $\int_0^{2\pi} f(x)dx = 1$ as well as $\int_0^{2\pi} xf(x)dx = 1$.

During the course it was demonstrated that $\{g_0, g_1, h_1, g_2, h_2, ...\}$, (with $g_0(x) = \frac{1}{\sqrt{2\pi}}$, $g_k(x) = \frac{1}{\sqrt{\pi}}\cos(kx)$, $h_k(x) = \frac{1}{\sqrt{\pi}}\sin(kx)$) is a maximal orthonormal system (MOS) for $L^2(0, 2\pi)$.

(d) Give a MOS for the subspace D.

3. By $l^2_{\mathbb{C}}$ we denote the normed complex vector space of all sequences $\underline{a} = (a_1, a_2, ...)$ of complex numbers for $\|\underline{a}\| = \left(\sum_{n=1}^{\infty} |a_n|^2\right)^{1/2}$ is finite. Let $A : l^2_{\mathbb{C}} \longrightarrow l^2_{\mathbb{C}}$ be linear and bounded.

   (a) Prove : if $A : l^2_{\mathbb{C}} \longrightarrow l^2_{\mathbb{C}}$ is compact, then the adjoint A^* is also compact.

   (b) Suppose $A(\underline{a}) = (\lambda_1 a_1, \lambda_2 a_2, ...)$ is a 'multiplication' operator with $\lambda_n \in \mathbb{C}$. Prove: A is compact if and only if $\lim_{n\to\infty} \lambda_n = 0$.

   (c) Give an example of a non-compact A for which A^2 is compact.

4. The real linear space l^∞ of bounded sequences is normed by $\|(a_1, a_2, ...)\| = \sup_{n\in\mathbb{N}} |a_n|$. Define the 'Hardy' action $A(a_1, a_2, a_3, ...) = \left(a_1, \frac{a_1+a_2}{2}, \frac{a_1+a_2+a_3}{3}, ...\right)$ so $A(\underline{a})_n = \frac{a_1+a_2+...+a_n}{n}$.

   (a) Show that A defines a linear operator $l^\infty \longrightarrow l^\infty$.

   (b) Is A bounded or unbounded?

5. Let C denote the complex Banach space of all continuous complex valued functions on $[0, 1]$, equipped with $\|f\|_\infty = \sup_{0\le t\le 1} |f(t)|$. The linear map $A : C \longrightarrow C$ is defined by $Af(x) = \int_0^x (x-y)f(y)dy$ for all $f \in C$ anf $0 \le x \le 1$.

   (a) Show that A is bounded.

   (b) Show that $|A^n f(x)| \le \frac{\|f\|_\infty}{n!} x^n$ for all $f \in C$, $n \in \mathbb{N}$ and $x \in [0, 1]$.

   (c) Prove that $(Id - A)$ is invertible.

   (d) Determine the spectrum of A.

GRADING POINTS

Total: 36+4=40 points

1.(a)2	2.(a)1	3.(a)2	4.(a)2	5.(a)2
(b)2	(b)2	(b)<u>3</u>	(b)2	(b)2
(c)2	(c)2	(c)<u>3</u>		(c)2
(d)2	(d)2			(d)<u>3</u>