Course : **Applied Functional Analysis (191506302)**

Date : **Tuesday, January 28, 2014**
Time : **08:45 - 11:45**

- An explanation to every answer is required
- You can make use of a calculator
- (Part of) the score to excercises 3 and 6 may be earned by homeworks; see the score table at the end

1. The operator $A : L^2(-\infty, \infty) \to L^2(-\infty, \infty)$ is defined by

$$(Af)(x) = f(2x) \text{ for all } x \in \mathbb{R}$$

Here $L^2(-\infty, \infty)$ is the space of classes of squared integrable functions

(a) Show that A is linear
(b) Determine $\|A\|$
(c) Find the adjoint of A

2. Let H be a Hilbert space with inner product $< , >$ and nulvector $\mathbf{0}$. A sequence $(\mathbf{x}_n)_n \subset H$ is said to *converge weakly* to a vector $\mathbf{x} \in H$ if

$$< \mathbf{x}_n, \mathbf{y} > \to < \mathbf{x}, \mathbf{y} > \text{ as } n \to \infty \text{ for all } \mathbf{y} \in H$$

In this case $\mathbf{x}$ is called a *weak limit* of the sequence $(\mathbf{x}_n)_n$

(a) Show that if $\mathbf{x}_n \to \mathbf{x}$ in norm, then $\mathbf{x}_n \to \mathbf{x}$ weakly
(b) Let $\mathbf{e}_n$ be the n-th unit vector of ℓ^2. Show that $\mathbf{e}_n \to \mathbf{0}$ weakly, but not in norm.
(c) Show that if $\mathbf{x}_n \to \mathbf{x}$ weakly and $\|\mathbf{x}_n\| \to \|\mathbf{x}\|$, then $\mathbf{x}_n \to \mathbf{x}$ in norm.
(d) Let $\mathbf{x}_n \to \mathbf{x}$ weakly and $T \in BL(H)$. Show that $T\mathbf{x}_n \to T\mathbf{x}$ weakly

3. As in our course $C[0,1]$ denotes the real vector space of continuous functions on $[0,1]$ with the maximum norm. And ℓ^∞ consists of all bounded (real) sequences with the supremum norm.

(a) Let $A : C[0,1] \to \ell^\infty$ the linear map defined by

$$Af = (f(1), f(\frac{1}{2}), f(\frac{1}{3}), f(\frac{1}{4}), \ldots)$$

What is the range of A?

(b) Give a vector $\neq \mathbf{0}$ in the null space of A.

(c) Does there exist a linear map $B : \ell^\infty \to C[0,1]$ with $\|B(\mathbf{x})\| = \|\mathbf{x}\|$ for all $\mathbf{x} \in \ell^\infty$. If so , define such a map; if not, motivate the non-existence.

4. The operator $T : C([0,1]) \to C([0,1])$ is defined by

$$Tf(x) = \int_{y=0}^{1} k(x,y) f(y) dy$$

where $k(x,y)$ is a continuous function with $\sup_{0 \leq x,y \leq 1} |k(x,y)| < 1$. Again $C[0,1]$ is endowed with max-norm.

(a) Prove that $\|T\| < 1$

(b) Show that the integral equation $f(x) - Tf(x) = g(x)$ has a unique solution f for every $g \in C[0,1]$.

(c) Solve the equation

$$f(x) - \int_0^1 \frac{1}{2} xy f(y) dy = x$$

5. We define the operate $A : \ell^2_{\mathbb{C}} \to \ell^2_{\mathbb{C}}$ by

$$A(a_1, a_2, a_3, \ldots) = (ia_2, -ia_1, \frac{1}{2} ia_4, -\frac{1}{2} ia_3, \frac{1}{3} ia_6, \ldots)$$

in formula $(A\mathrm{a})_{2n-1} = \frac{1}{n} ia_{2n}$ and $(A\mathrm{a})_{2n} = -\frac{1}{n} ia_{2n-1}$

(a) Prove that A is self-adjoint.

(b) Prove that A is compact.

(c) Determine the spectrum of $\sigma(A)$ of A.

6. Formulate the theorem of Hahn-Banach for a Banach space.

Grading points:

1.	(a)	2	2.	(a)	2	3.	(a)	2	4.	(a)	2	5.	(a)	2	6	3
	(b)	2		(b)	2		(b)	2		(b)	2		(b)	2		
	(c)	2		(c)	2		(c)	2		(c)	3		(c)	2		
				(d)	2											