Exam: Continuous Optimisation 2015

1. Let $f : \mathcal{C} \to \mathbb{R}$, $\mathcal{C} \subset \mathbb{R}^n$ convex, be a convex function. Show that then the following holds: [3 points]
   A local minimizer of f on $\mathcal{C}$ is a global minimizer on $\mathcal{C}$. And a strict local minimizer of f on $\mathcal{C}$ is a strict global minimizer on $\mathcal{C}$.

> **Solution:** for a local minimizer $\overline{\mathbf{x}}$: Suppose $\overline{\mathbf{x}}$ is not a global minimiser. Then with some $\mathbf{y} \in \mathcal{C}$ we have $f(\overline{\mathbf{x}}) > f(\mathbf{y})$. Thus for $0 < \lambda \leq 1$ we find with $\mathbf{x}_\lambda := \overline{\mathbf{x}} + \lambda(\mathbf{y} - \overline{\mathbf{x}})$ using convexity of f:
>
> $$f(\mathbf{x}_\lambda) \leq f(\overline{\mathbf{x}}) + \lambda[f(\mathbf{y}) - f(\overline{\mathbf{x}})] < f(\overline{\mathbf{x}})$$
>
> So letting $\lambda \to 0^+$, $\overline{\mathbf{x}}$ cannot be a local minimizer.
> for a strict local minimizer $\overline{\mathbf{x}}$: Suppose it is not a strict global minimiser. Then with some $\mathbf{y} \in \mathcal{C}, \overline{\mathbf{x}} \neq \mathbf{y}$ we have $f(\overline{\mathbf{x}}) \geq f(\mathbf{y})$. Thus for $0 < \lambda \leq 1$ we find with $\mathbf{x}_\lambda := \overline{\mathbf{x}} + \lambda(\mathbf{y} - \overline{\mathbf{x}})$ using convexity of f:
>
> $$f(\mathbf{x}_\lambda) \leq f(\overline{\mathbf{x}}) + \lambda[f(\mathbf{y}) - f(\overline{\mathbf{x}})] \leq f(\overline{\mathbf{x}})$$
>
> So letting $\lambda \to 0^+$, $\overline{\mathbf{x}}$ cannot be a strict local minimizer.

2. (a) Show that for $\mathbf{d} \in \mathbb{R}^n$ it holds: [2 points]

$$\mathbf{d}^\mathsf{T}\mathbf{x} \geq 0 \ \forall \mathbf{x} \in \mathbb{R}^n \quad \Leftrightarrow \quad \mathbf{d} = 0.$$

   (b) Let $\mathbf{c}, \mathbf{a}_i \in \mathbb{R}^n, i = 1, \ldots, m$ $(m \geq 1)$. Show using the Farkas Lemma (lecture sheets, Th. 5.1) that precisely one of the following alternatives (I) or (II) is true: [3 points]
   (I): $\mathbf{c}^\mathsf{T}\mathbf{x} < 0, \ \mathbf{a}_i^\mathsf{T}\mathbf{x} \leq 0, i = 1, \ldots, m$ has a solution $\mathbf{x} \in \mathbb{R}^n$.
   (II): there exist $\mu_1 \geq 0, \ldots, \mu_m \geq 0$ such that: $\mathbf{c} + \sum_{i=1}^m \mu_i \mathbf{a}_i = 0$

> **Solution:**
>
> (a) "$\Rightarrow$":
>
> $$\mathbf{d}^\mathsf{T}\mathbf{x} \geq 0 \ \forall \mathbf{x} \in \mathbb{R}^n \Rightarrow \pm\mathbf{d}^\mathsf{T}\mathbf{e}_j \geq 0 \forall j \Rightarrow \mathbf{d}^\mathsf{T}\mathbf{e}_j = 0 \ \forall j \Rightarrow \mathbf{d} = \mathbf{0}$$
>
> "$\Leftarrow$":
>
> $$\mathbf{d} = \mathbf{0} \Rightarrow \mathbf{d}^\mathsf{T}\mathbf{x} = 0 \ \forall \mathbf{x} \in \mathbb{R}^n \Rightarrow \mathbf{d}^\mathsf{T}\mathbf{x} \geq 0 \ \forall \mathbf{x} \in \mathbb{R}^n$$
>
> (b) Considering $\mathbf{a}_{m+1} = \mathbf{c}$ and $\mathbf{b} = -\mathbf{e}_{m+1} \in \mathbb{R}^{m+1}$ we have that (I) is equivalent to:
>
> (i): $\mathbf{a}_i^\mathsf{T}\mathbf{x} \leq b_i$, $i = 1, \ldots, (m+1)$ has a solution $\mathbf{x}$.
>
> By Farkas' Lemma, precisely one of either (i) or the following statement,

(ii), is true:

(ii): $\exists \mathbf{y} \in \mathbb{R}_+^{m+1}$ such that $\mathbf{0} = \sum_{i=1}^{m+1} y_i \mathbf{a}_i$, $0 > \mathbf{b}^\mathsf{T}\mathbf{y}$.

This is equivalent to:

$\exists \mathbf{y} \in \mathbb{R}_+^{m+1}$ such that $\mathbf{0} = y_{m+1}\mathbf{c} + \sum_{i=1}^{m} y_i \mathbf{a}_i$, $0 > -y_{m+1}$,

which in turn is equivalent to (II).

3. Given is the problem

$$(P) \qquad \min_{\mathbf{x}\in\mathbb{R}^2} \; (-2x_1 - x_2) \qquad \text{s.t.} \qquad x_1 \le 0, \text{ and } -(x_1-1)^2 - (x_2-1)^2 + 2 \le 0 \,.$$

(a) Is (P) a convex problem? Sketch the feasible set and the level set of f given by $f(\mathbf{x}) = f(\overline{\mathbf{x}})$ with $\overline{\mathbf{x}} = 0$. Is LICQ (constraint qualification) satisfied at $\overline{\mathbf{x}}$? [3 points]

(b) Show that the point $\overline{\mathbf{x}} = 0$ is a KKT-point of (P). Determine the corresponding Lagrangean multipliers. [3 points]

(c) Show that $\overline{\mathbf{x}}$ is a local minimizer. What is the order of this minimizer? Is it a global minimizer? [3 points]

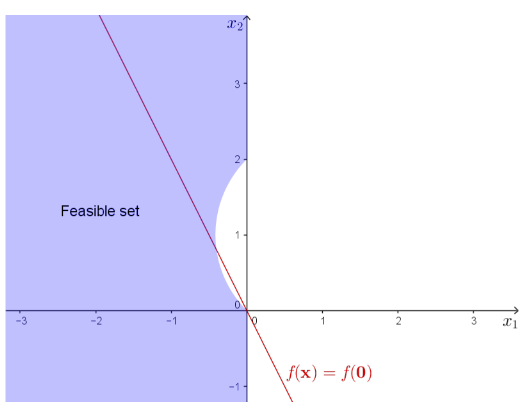
(d) Consider now the program (objective f and constraint function g_2 interchanged): [2 points]

$$(\widetilde{P}) \qquad \min_{\mathbf{x}\in\mathbb{R}^2} -(x_1-1)^2 - (x_2-1)^2 + 2 \qquad \text{s.t.} \qquad x_1 \le 0, \text{ and } -2x_1 - x_2 \le 0 \,.$$

Explain (without any further calculations) why $\overline{\mathbf{x}} = 0$ is also a local minimizer of $(\widetilde{P})$.

Solution:

(a) (P) is not a convex program since g_2 is not convex: $\nabla^2 g_2(\mathbf{x}) = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$ is negative definite.



Above is a sketch of the problem. The feasible set is coloured blue and the level curve is coloured red.

LICQ holds at $\overline{\mathbf{x}} = \mathbf{0}$:

$$\nabla g_1(\mathbf{x}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \nabla g_2(\overline{\mathbf{x}}) = \begin{pmatrix} 2 \\ 2 \end{pmatrix} \quad \text{are linearly independent}$$

Give a complete sketch.

(b) The KKT condition for $\overline{\mathbf{x}} = \mathbf{0}$ (g_1 and g_2 active) read:

$$\begin{pmatrix} -2 \\ -1 \end{pmatrix} + \mu_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \mu_2 \begin{pmatrix} 2 \\ 2 \end{pmatrix} = 0$$

With (unique) solution $\mu_1 = 1, \mu_2 = 1/2$.

(c) Since the assumptions of Th 5.13 are satisfied, $\overline{\mathbf{x}} = \mathbf{0}$ is a local minimizer of order $p = 1$.
It is not a global minimizer since $f(\overline{\mathbf{x}}) = 0$ and e.g. for feasible $x = (0, x_2),\ x_2 \geq 2$ we have $f(0, x_2) \to -\infty$ for $x_2 \to \infty$.

(d) The KKT condition at $\overline{\mathbf{x}} = \mathbf{0}$ for (P) directly yields a corresponding KKT condition for $(\widetilde{P})$ at $\overline{\mathbf{x}}$ (feasible for $(\widetilde{P})$!!) which again satisfies the assumption of Theorem 5.13 for $(\widetilde{P})$.

4. Consider the (nonlinear) program: [3 points]

$$(P) \qquad \min_{\mathbf{x}} \ f(\mathbf{x}) \qquad \text{s.t.} \quad \mathbf{x} \in \ \mathcal{F} := \{\mathbf{x} \in \mathbb{R}^n \mid g_j(\mathbf{x}) \leq 0,\ j \in J\}$$

with $f, g_j \in C^1$, $f, g : \mathbb{R}^n \to \mathbb{R}$, $J = \{1, \ldots, m\}$. Let $\mathbf{d}_k$ be a strictly feasible descent direction for $\mathbf{x}_k \in \mathcal{F}$. Show that for $t > 0$, small enough, it holds:

$$f(\mathbf{x}_k + t\mathbf{d}_k) < f(\mathbf{x}_k) \quad \text{and} \quad \mathbf{x}_k + t\mathbf{d}_k \in \mathcal{F}$$

Solution: By using Taylor around $\mathbf{x}_k$ we find for $j \in J_{\mathbf{x}_k}$ (use $\nabla g_j(\mathbf{x}_k)^\mathsf{T}\mathbf{d}_k < 0;\ g_j(\mathbf{x}_k) = 0$):

$$g_j(\mathbf{x}_k + t\mathbf{d}_k) = g_j(\mathbf{x}_k) + t\nabla g_j(\mathbf{x}_k)^\mathsf{T}\mathbf{d}_k + o(t) = t\nabla g_j(\mathbf{x}_k)^\mathsf{T}\mathbf{d}_k + o(t) < 0 \quad \text{for } t > 0 \text{ small enough.}$$

By continuity also for $j \notin J_{\mathbf{x}_k}$ we have $g_j(\mathbf{x}_k + t\mathbf{d}_k) < 0$ for $t > 0$ small enough.
So $\mathbf{x}_k + t\mathbf{d}_k \in \mathcal{F}$. In view of $\nabla f(\mathbf{x}_k)^\mathsf{T}\mathbf{d}_k < 0$ we also find

$$f(\mathbf{x}_k + t\mathbf{d}_k) = f(\mathbf{x}_k) + t\nabla f(\mathbf{x}_k)^\mathsf{T}\mathbf{d}_k + o(t) < f(\mathbf{x}_k) \quad \text{for } t > 0 \text{ small enough.}$$

5. For a given nonempty set $\mathcal{A} \subseteq \mathbb{R}^n$ we define its conic hull, conic($\mathcal{A}$) by

$$\operatorname{conic}(\mathcal{A}) := \left\{ \sum_{i=1}^{m} \mu^i \mathbf{x}^i \;:\; \mathbf{x}^i \in \mathcal{A},\ \mu^i \geq 0 \text{ for all } i,\ m \in \mathbb{N} \right\}.$$

(a) Show that conic($\mathcal{A}$) is a convex cone. [2 points]

(b) Show that if $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathbb{R}^n$, with $\mathcal{B}$ being a convex cone, then $\operatorname{conic}(\mathcal{A}) \subseteq \mathcal{B}$. [3 points]

(c) Show that conic($\mathcal{A}$) is full dimensional if and only if there does not exist $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathcal{A}$. [1 point]

Solution:

(a) By Theorem 7.2, equivalently we want to show that for all $\mathbf{u}, \mathbf{v} \in \operatorname{conic}(\mathcal{A})$ and $\lambda_1, \lambda_2 > 0$ we have $\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} \in \operatorname{conic}(\mathcal{A})$.

Considering an arbitrary $\mathbf{u}, \mathbf{v} \in \operatorname{conic}(\mathcal{A})$ and $\lambda_1, \lambda_2 > 0$ we have

$$\mathbf{u} = \sum_{i=1}^{m} \mu^i \mathbf{x}^i, \qquad \mathbf{v} = \sum_{i=1}^{p} \nu^i \mathbf{y}^i,$$

$$\begin{aligned} \text{for some } & \mathbf{x}^1, \ldots, \mathbf{x}^m, \mathbf{y}^1, \ldots, \mathbf{y}^p \in \mathcal{A}, \\ & \mu^1, \ldots, \mu^m, \nu^1, \ldots, \nu^p \geq 0, \\ & p, m \in \mathbb{N}. \end{aligned}$$

Therefore

$$\lambda_1 \mathbf{u} + \lambda_2 \mathbf{v} = \sum_{i=1}^{m} \underbrace{\lambda_1 \mu^i}_{\geq 0} \mathbf{x}^i + \sum_{i=1}^{p} \underbrace{\lambda_2 \nu^i}_{\geq 0} \mathbf{y}^i \in \operatorname{conic}(\mathcal{A}).$$

(b) For $k \in \mathbb{N}$, let $\mathcal{L}^k := \left\{ \sum_{i=1}^{k} \mu^i \mathbf{x}^i \;:\; \mathbf{x}^i \in \mathcal{A},\ \mu^i \geq 0 \text{ for all } i \right\}$. We will prove by induction that $\mathcal{L}^k \subseteq \mathcal{B}$ for all $k \in \mathbb{N}$, and thus $\mathcal{B} \supseteq \bigcup_{k \in \mathbb{N}} \mathcal{L}^k = \operatorname{conic}(\mathcal{A})$.

We start by proving the case of $k = 1$. If $\mathbf{y} \in \mathcal{L}^1$ then $\mathbf{y} = \mu \mathbf{x}$ for some $\mu \geq 0$ and $\mathbf{x} \in \mathcal{A}$. We thus have $\mathbf{x} \in \mathcal{B}$, and as $\mathcal{B}$ is a cone we have $\mathbf{y} = \mu \mathbf{x} \in \mathcal{B}$.

We now suppose the statement is true for k, and show it is also true for $k+1$. If $\mathbf{y} \in \mathcal{L}^{k+1}$ then $\mathbf{y} = \sum_{i=1}^{k+1} \mu^i \mathbf{x}^i$ where $\mathbf{x}^i \in \mathcal{A}$ and $\mu^i \geq 0$ for all i. Letting $\mathbf{z}^1 = \sum_{i=1}^{k} 2\mu^i \mathbf{x}^i \in \mathcal{L}^k \subseteq \mathcal{B}$ and $\mathbf{z}^2 = 2\mu^{k+1} \mathbf{x}^{k+1} \in \mathcal{L}^1 \subseteq \mathcal{B}$, the set $\mathcal{B}$ being convex implies that $\mathcal{B} \ni \frac{1}{2}\mathbf{z}^1 + \frac{1}{2}\mathbf{z}^2 = \mathbf{y}$.

Alternatively:

$$\begin{aligned}
\operatorname{conic}(\mathcal{A}) &= \left\{ \sum_{i=1}^{m} \mu^i \mathbf{x}^i \ : \ \mathbf{x}^i \in \mathcal{A},\ \mu^i \geq 0 \text{ for all } i,\ m \in \mathbb{N} \right\} \\
&= \{\mathbf{0}\} \cup \left\{ \sum_{i=1}^{m} \mu^i \mathbf{x}^i \ : \ \mathbf{x}^i \in \mathcal{A},\ \mu^i \geq 0 \text{ for all } i,\ m \in \mathbb{N},\ \lambda = \sum_{i=1}^{m} \mu^i > 0 \right\} \\
&= \{\mathbf{0}\} \cup \left\{ \lambda \sum_{i=1}^{m} \theta^i \mathbf{x}^i \ : \ \mathbf{x}^i \in \mathcal{A},\ \theta^i \geq 0 \text{ for all } i,\ m \in \mathbb{N},\ 1 = \sum_{i=1}^{m} \theta^i,\ \lambda > 0 \right\} \\
&= \{\mathbf{0}\} \cup \mathbb{R}_{++} \operatorname{conv}(\mathcal{A}) = \mathbb{R}_+ \operatorname{conv}(\mathcal{A}).
\end{aligned}$$

As $\mathcal{B}$ is convex, we have $\operatorname{conv}(\mathcal{A}) \subseteq \mathcal{B}$. As $\mathcal{B}$ is a cone we then get

$$\mathcal{B} \supseteq \mathbb{R}_+ \operatorname{conv}(\mathcal{A}) = \operatorname{conic}(\mathcal{A}).$$

(c) We will prove the equivalent statement that $\operatorname{conic}(\mathcal{A})$ is not full dimensional if and only if there exists $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathcal{A}$.

($\Rightarrow$) Suppose $\operatorname{conic}(\mathcal{A})$ is not full-dimensional. Then by definition 7.8.3 there exists $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \operatorname{conic} \mathcal{A}$. We trivially have $\mathcal{A} \subseteq \operatorname{conic}(\mathcal{A})$ and thus $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathcal{A}$.

($\Leftarrow$) Suppose there exists $\mathbf{y} \in \mathbb{R}^n \setminus \{\mathbf{0}\}$ such that $\langle \mathbf{y}, \mathbf{x} \rangle = 0$ for all $\mathbf{x} \in \mathcal{A}$. Then for all $\mathbf{z} \in \operatorname{conic}(\mathcal{A})$ we have $\mathbf{z} = \sum_{i=1}^{m} \mu^i \mathbf{x}^i$ for some $\mathbf{x}^i \in \mathcal{A}$ and $\mu^i \geq 0$ for all i, $m \in \mathbb{N}$, and thus $\langle \mathbf{y}, \mathbf{z} \rangle = \sum_{i=1}^{m} \mu^i \langle \mathbf{y}, \mathbf{x}^i \rangle = 0$. Therefore, by definition 7.8.3, we have that $\operatorname{conic}(\mathcal{A})$ is not full-dimensional.

6. In this question we will consider the proper cone $\mathcal{K} \subseteq \mathbb{R}^{n+2}$ defined as

$$\mathcal{K} = \left\{ \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} \ : \ \mathbf{y} \in \mathbb{R}^n,\ x, z \in \mathbb{R},\ \|\mathbf{y}\|_2 \leq x,\ z \geq 0 \right\}.$$

(a) Consider a ray $\mathcal{R} = \{\mathbf{c} - y_1 \mathbf{a} \mid y_1 \in \mathbb{R}_+\}$ with fixed $\mathbf{a}, \mathbf{c} \in \mathbb{R}^n$. We wish to find the distance between the origin and the closest point in this ray. Formulate this problem as a conic optimisation problem over $\mathcal{K}$. [2 points]

(b) Give an explicit characterisation of $\mathcal{K}^*$. [1 point]
[Justification for your answer must be provided]

(c) What is the dual problem to your formulation in part (a)? [2 points]

[If you were not able to answer parts (a) and (b) then instead find the dual to: $\min_y \quad y \quad \text{s.t.} \quad \mathbf{c} + y\mathbf{a} \in \mathbb{R}^n_+$.]

Solution:

(a) This problem is equivalent to the following problems

$$\min_{y_1} \quad \|\mathbf{c} - y_1\mathbf{a}\|_2 \quad \text{s.t.} \quad y_1 \geq 0,$$

$$\begin{aligned} \min_{\mathbf{y}} \quad & y_2 \\ \text{s.t.} \quad & \|\mathbf{c} - y_1\mathbf{a}\|_2 \leq y_2, \qquad y_1 \geq 0, \end{aligned}$$

$$\begin{aligned} \min_{\mathbf{y}} \quad & y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 0 \\ \mathbf{c} \\ 0 \end{pmatrix} - y_1 \begin{pmatrix} 0 \\ \mathbf{a} \\ -1 \end{pmatrix} - y_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{K} \end{aligned}$$

$$\begin{aligned} -\max_{\mathbf{y}} \quad & 0y_1 - y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 0 \\ \mathbf{c} \\ 0 \end{pmatrix} - y_1 \begin{pmatrix} 0 \\ \mathbf{a} \\ -1 \end{pmatrix} - y_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{K} \end{aligned}$$

The correct answer is either of the last two formulations, or equivalent.

(b) We have that $\mathcal{K} = \mathcal{L}_n \times \mathbb{R}_+$, and thus $\mathcal{K}^* = \mathcal{L}_n^* \times \mathbb{R}_+^* = \mathcal{L}_n \times \mathbb{R}_+ = \mathcal{K}$.

(c) Considering

$$\begin{aligned} -\max_{\mathbf{y}} \quad & 0y_1 - y_2 \\ \text{s.t.} \quad & \begin{pmatrix} 0 \\ \mathbf{c} \\ 0 \end{pmatrix} - y_1 \begin{pmatrix} 0 \\ \mathbf{a} \\ -1 \end{pmatrix} - y_2 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} \in \mathcal{K} \end{aligned}$$

the dual problem is

$$\begin{aligned} -\min_{x,\mathbf{y},z} \quad & \left\langle \begin{pmatrix} 0 \\ \mathbf{c} \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} \right\rangle \\ \text{s.t.} \quad & \left\langle \begin{pmatrix} 0 \\ \mathbf{a} \\ -1 \end{pmatrix}, \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} \right\rangle = 0 \\ & \left\langle \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} \right\rangle = -1, \\ & \begin{pmatrix} x \\ \mathbf{y} \\ z \end{pmatrix} \in \mathcal{K}^* \end{aligned}$$

This can be simplified to

$$\begin{aligned} \max_{x,\mathbf{y},z} \quad & -\langle \mathbf{c}, \mathbf{y} \rangle \\ \text{s.t.} \quad & z = \langle \mathbf{a}, \mathbf{y} \rangle \\ & x = 1, \quad z \geq 0, \quad \|\mathbf{y}\|_2 \leq x \end{aligned}$$

which in turn is equivalent to

$$\max_{\mathbf{y}} \quad \langle -\mathbf{c}, \mathbf{y} \rangle \qquad \text{s.t.} \qquad \langle \mathbf{a}, \mathbf{y} \rangle \geq 0, \quad \|\mathbf{y}\|_2 \leq 1.$$

Alternative question:

The problem is equivalent to $-\max_y \quad -y \quad \text{s.t.} \quad \mathbf{c} - y(-\mathbf{a}) \in \mathbb{R}^n_+$.

The dual to this is $-\min_{\mathbf{x}} \quad \langle \mathbf{c}, \mathbf{x} \rangle \quad \text{s.t.} \quad \langle -\mathbf{a}, \mathbf{x} \rangle = -1, \quad \mathbf{x} \in \mathbb{R}^n_+$,

which is equivalent to $\max_{\mathbf{x}} \quad \langle -\mathbf{c}, \mathbf{x} \rangle \quad \text{s.t.} \quad \langle \mathbf{a}, \mathbf{x} \rangle = 1, \quad \mathbf{x} \in \mathbb{R}^n_+$

7. Consider the following optimisation problem: [3 points]

$$\begin{aligned} \min_{\mathbf{x}} \quad & 2x_2^2 + 5x_1x_2 - 4x_2 \\ \text{s.t.} \quad & 2x_1^2 + x_1 + 3x_2^2 - 2x_1x_2 = 3 \\ & \mathbf{x} \in \mathbb{R}^2. \end{aligned} \tag{A}$$

Give the standard positive semidefinite approximation for this problem, the solution of which would provide a lower bound to the optimal value of problem (A).

Solution: This problem is equivalent to

$$\begin{aligned}
\min_{\mathbf{x}} \quad & \left\langle \begin{pmatrix} 0 & 5/2 \\ 5/2 & 2 \end{pmatrix}, \mathbf{x}\mathbf{x}^\mathsf{T} \right\rangle - 4x_2 \\
\text{s. t.} \quad & \left\langle \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, \mathbf{x}\mathbf{x}^\mathsf{T} \right\rangle + x_1 = 3 \\
& \begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathbf{x}\mathbf{x}^\mathsf{T} \end{pmatrix} \in \mathcal{PSD}^3 \\
& \mathbf{x} \in \mathbb{R}^3,
\end{aligned}$$

which can be relaxed to

$$\begin{aligned}
\min_{\mathbf{x}, \mathsf{X}} \quad & \left\langle \begin{pmatrix} 0 & 5/2 \\ 5/2 & 2 \end{pmatrix}, \mathsf{X} \right\rangle - 4x_2 \\
\text{s. t.} \quad & \left\langle \begin{pmatrix} 2 & -1 \\ -1 & 3 \end{pmatrix}, \mathsf{X} \right\rangle + x_1 = 3 \\
& \begin{pmatrix} 1 & \mathbf{x}^\mathsf{T} \\ \mathbf{x} & \mathsf{X} \end{pmatrix} \in \mathcal{PSD}^3 \\
& \mathbf{x} \in \mathbb{R}^3
\end{aligned}$$

8. (Automatic additional points) [4 points]

Question:	1	2	3	4	5	6	7	8	Total
Points:	3	5	11	3	6	5	3	4	40

A copy of the lecture-sheets may be used during the examination.
Good luck!