Re-exam Continuous Optimization

28 February 2022, 14.00–17.00

This closed-book exam consists of 5 questions. Please start each question on a new page, write legibly, and try to hand in your work with the solutions in the correct order. Good luck!

1. Consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & -\log(x) \\ \text{subject to} \quad & x \leq 100. \end{aligned}$$

   (a) (3 points) Show this is a convex problem.

   (b) (4 points) Derive the dual problem and compute an optimal solution of the dual.

2. Given a graph G on the vertices $\{1, \ldots, n\}$, the Lovász theta number $\vartheta(G)$ is the optimal value of the optimization problem

$$\begin{aligned} \text{maximize} \quad & \langle J, X \rangle \\ \text{subject to} \quad & X \succcurlyeq_{S^n_+} 0, \\ & \langle I, X \rangle = 1, \\ & X_{ij} = 0 \text{ if } i \sim j. \end{aligned}$$

   (a) (2 points) Explain why the above optimization is a conic program.

   (b) (3 points) Recall that an independent set is a subset of the vertices no two of which are adjacent, and $\alpha(G)$ is the size of a largest independent set. Show $\vartheta(G) \geq \alpha(G)$.

   (c) (8 points) Derive the Lagrange dual of the above optimization problem.

3. (5 points) Consider the function

$$F(x, y) = \sin(x + y)x$$

   where we view addition, multiplication, and the sine function as elementary functions. Show how $\nabla F(\pi, \pi)$ is computed using reverse-mode automatic differentiation by drawing the appropriate diagrams.

4. (a) (5 points) Let f be a strictly convex, differentiable function. Explain when and why gradient descent with exact line search applied to f converges very slowly. Why is this not in contradiction with the fact that gradient descent with exact line search converges linearly?

   (b) (5 points) Suppose we want to find a local minimum of a nonconvex, twice continuously differentiable function. Explain what can go wrong when applying Newton's method. How can we solve this?

5. For $u \in \mathbb{R}^m$ we consider the problem

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \leq u, \quad i = 1, \ldots, m \end{array}$$

with optimal value $p^*(u)$. Assume strong duality holds and that for $u = 0$ the dual has an optimal solution λ^*.

(a) (7 points) Show that the global sensitivity inequality holds: For all $u \in \mathbb{R}^m$ we have

$$p^*(u) \geq p^*(0) - u^\mathsf{T}\lambda^*.$$

(b) (3 points) What does $\lambda_i^* = 0$ mean for the nonperturbed ($u = 0$) problem?

End of test