Exam Continuous Optimization

19 January 2026, 10.00–13.00

This closed-book exam consists of 5 questions. Please start each question on a new A4 sheet, write legibly, and hand in your work with the solutions in the correct order. In total, you can obtain 90 points. The final grade is $1 + \#\text{points}/10$ rounded to half integers (except 5.5). Good luck!

1. Let W be a positive definite matrix and consider the optimization problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2} x^\mathsf{T} W x \\ \text{subject to} \quad & Ax = b. \end{aligned}$$

   (a) (5 points) Show this is a convex optimization problem.

   (b) (15 points) Derive the Lagrangian, Lagrange dual function, and Lagrange dual problem.

   (c) (5 points) Show that for any starting point x_0 satisfying $Ax_0 = b$, Newton's method converges to the minimizer in a single iteration when applied to this equality constrained problem.

2. Suppose we apply the golden section search method to a unimodal function $f\colon \mathbb{R} \to \mathbb{R}$ with an initial bracket $[a, b]$. At each iteration, the function is evaluated at the points

$$x_1 = \frac{1}{\varphi} a + \left(1 - \frac{1}{\varphi}\right) b \quad \text{and} \quad x_2 = \left(1 - \frac{1}{\varphi}\right) a + \frac{1}{\varphi} b,$$

   where $\varphi \approx 1.618$ denotes the golden ratio. These evaluations are then used to update the bracket.

   (a) (5 points) Describe how the interval $[a, b]$ is updated at each iteration, and explain why the updated interval is guaranteed to contain the minimizer of f.

   (b) (10 points) What is the order and rate of convergence of golden section search? Provide a full justification of your answer referring to the definition of the convergence order/rate.

3. (15 points) Consider the function

$$F(x_1, x_2) = \frac{x_1^2 + x_2^2}{\cos(x_2)}$$

   where we view addition, taking the square, the cosine, and division as elementary functions. Compute $\nabla F(3, 0)$ using reverse-mode automatic differentiation by drawing the appropriate diagrams.

4. Consider the primal-dual interior point method as discussed in class with the parameter $\mu = 2/3$. We apply this to the problem

$$\begin{aligned} \text{minimize} \quad & x_1 + x_2 \\ \text{subject to} \quad & 3x_1^4 + x_2^2 \leq 10. \end{aligned}$$

Suppose that the current primal-dual iterate is (x, λ) with $x = (1, 1)$ and $\lambda = 1$.

(a) (5 points) Compute the surrogate duality gap $\widehat{\eta}(x, \lambda)$. Explain why this implies x and λ as given above cannot both be optimal.

(b) (10 points) Derive the concrete linear system one needs to solve to compute the primal-dual search direction $(\Delta x, \Delta\lambda)$ starting from the current iterate as given above.

5. (a) (5 points) The ADMM update steps for a problem of the form

$$\begin{aligned} \text{minimize} \quad & f(x) + g(y) \\ \text{subject to} \quad & Ax + By = b, \end{aligned}$$

are

$$\begin{aligned} x_{k+1} &= \operatorname{argmin}_x L_\rho(x, y_k, \nu_k), \\ y_{k+1} &= \operatorname{argmin}_y L_\rho(x_{k+1}, y, \nu_k), \\ \nu_{k+1} &= \nu_k + \rho(Ax_{k+1} + By_{k+1} - b), \end{aligned}$$

where L_ρ is the augmented Lagrangian. Derive from this the scaled ADMM update steps with dual variable

$$z_k = \frac{1}{\rho}\nu_k.$$

(b) (15 points) Let C be a nonempty closed convex set and consider the problem

$$\begin{aligned} \text{minimize} \quad & \frac{1}{2}\|Ax - b\|_2^2 \\ \text{subject to} \quad & x \in C. \end{aligned}$$

Write this in a form so that you can apply ADMM, and give the scaled ADMM update steps. Simplify the update steps as much as possible.

End of test