Exam MSc Course Game Theory, University of Twente (191521800)

November 6, 2014

Please motivate all your answers. Note: This exam comes with a cheat sheet that contains most of the basic definitions. (See the last two pages.) Other necessary definitions are given in the questions below.

Questions:

1. (3 points) Please give a brief argument or a counterexample to prove or falsify each of the following statements.

   (a) In a sealed bid second price auction, if I know that other bidders are not truthful, I might gain from not being truthful, too.

   (b) In a sealed bid first price auction, assume I happen to know all other bids. Then my utility depends linearly on my bid b_i.

   (c) In a 2×2 strategic form game, if there is a pure strategy Nash equilibrium, either at least one player has a dominant strategy, or there are two pure Nash equilibria.

2. (5 points) Consider the (symmetric) bimatrix game given by

$$(A, B) = \begin{pmatrix} -10, -10 & 0, 5 \\ 5, 0 & -1, -1 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game, and give a correlated equilibrium that is not a Nash equilibrium.

3. (6 points) Consider the following two player extensive form game.

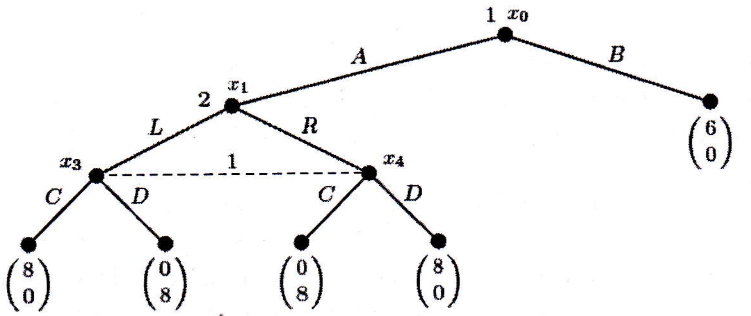


Figure 1: 2-player extensive form game.

   (a) Give the corresponding 3×2 bimatrix game.

   (b) Compute all Nash equilibria of the corresponding 3×2 bimatrix game, and also give the corresponding behavioral strategy.

   (c) Compute subgame perfect equilibria for this game, and briefly discuss the outcome.

4. (7 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	2	5	14	15	10	20

(a) Is the game a convex game?

(b) Compute the core $C(N, v)$, and the domination core $DC(N, v)$. Are they equal?

(c) What is the maximal value of $v(\{2\})$ such that the core still is nonempty?

5. (5 points)

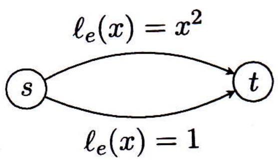
   (a) Let $(\{1, 2, 3\}, v)$ be a three-person game which has a nonempty core $C \neq \emptyset$. Show that $2v(\{1, 2, 3\}) \geq v(\{1, 2\}) + v(\{1, 3\}) + v(\{2, 3\})$.

   (b) Prove that a game (N, v) is convex if and only if

   $$v(T) = \min_{\sigma \in \Pi} \sum_{i \in T} m_i^\sigma(v)$$

   for all coalitions T, $T \neq \emptyset$. Here, Π denotes the set of all permutations σ of the player set N. (Hint: recall that a game is convex if and only if the core C is equal to the Weber set W.)

6. (5 points) Here we show that the price of anarchy can be unbounded when the latency functions are arbitrary instead of linear. First, consider the simple road network with nonlinear latency functions as shown below.



   (a) Assume that one unit of flow is to be sent from s to t, and it can be splitted arbitrarily. Show that the price of anarchy in this nonatomic routing instance exceeds 3/2 (Hint: $1/\sqrt{3} \approx 0.57735$).

   (b) Now adapt the latency functions and argue that, in general, the price of anarchy can become arbitrarily large (Hint: Depending on the latency functions you use, it may help to know that $(n+1)^{-1/n} \to 1$ for $n \to \infty$).

7. (5 points) Consider the following load balancing game. There are n tasks with processing times $p_j = 1$, $j = 1, \ldots, n$, that need to be distributed over $m < n$ machines. Machines have speeds $s_i \in (0, 1]$ (and we assume for simplicity that $1/s_i \in \mathbb{N}$). The latency experienced by any task equals the load of the machine it is processed on, divided by the speed of the machine. That is, let N_i be the tasks on machine i, the latency of all tasks in N_i equals $1/s_i \sum_{j \in N_i} p_j = |N_i|/s_i$.

   Tasks are interested in a latency as small as possible. Assume that tasks may selfishly select the machine to be processed on. Like in network routing games, we are interested in minimizing total latency of all tasks.

   (a) Show that this game always has a pure strategy Nash equilibrium. (Hint: Define a suitable potential function.)

   (b) Show that the price of anarchy (for pure strategy Nash equilibria) can be > 1 (as large as 4/3).

Total: 36 + 4 = 40 points

Basic definitions for MSc course on Game Theory

Noncooperative Game Theory

- Matrix games $A \in \mathbb{R}^{m\times n}$
  - Payoff row player $\mathbf{p}A\mathbf{q}$ with $\mathbf{p}$ = mixed strategy row player and $\mathbf{q}$ = mixed strategy column player. payoff column player $-\mathbf{p}A\mathbf{q}$.
  - Maximin strategy $\mathbf{p}$ for row player achieves maximum in $\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q}$. Minimax strategy $\mathbf{q}$ column player achieves minimum in $\min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}$.
  - von Neumann Theorem: $\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q} = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}$ for all (finite) A.
  - Entry (i,j) is saddlepoint if $a_{ij} \geq a_{kj}$ for all $k = 1,\ldots,m$ and $a_{ij} \leq a_{ik}$ for all $k = 1,\ldots,n$.
- Bimatrix Games (A,B) both $\in \mathbb{R}^{m\times n}$
  - Payoff row player $\mathbf{p}A\mathbf{q}$ with $\mathbf{p}$ = mixed strategy row player and $\mathbf{q}$ = mixed strategy column player, payoff column player $\mathbf{p}B\mathbf{q}$.
  - Carrier of strategy $\mathbf{p}$ for row player is $C(\mathbf{p}) = \{i \in \{1,\ldots,m\} : p_i > 0\}$, likewise for column player $C(\mathbf{q}) = \{j \in \{1,\ldots,n\} : q_j > 0\}$.
  - Nash equilibrium: strategy pair $(\mathbf{p},\mathbf{q})$ such that $\mathbf{p}A\mathbf{q} \geq \mathbf{p}'A\mathbf{q}$ for all $\mathbf{p}'$ and $\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{q}'$ for all $\mathbf{q}'$.
  - Equilibrium principle: $(\mathbf{p},\mathbf{q})$ is Nash equilibrium if and only if the pure strategies $C(\mathbf{p})$ are best replies to $\mathbf{q}$ and the pure strategies $C(\mathbf{q})$ are best replies to $\mathbf{p}$.
- Finite games $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$
  - N= set of n players, S_i = set of pure strategies of player i, $S = S_1 \times \cdots \times S_n$ = set of pure strategy profiles = set of possible outcomes, σ_i = mixed strategy of player i, $u_i(s) = u_i(s_1,\ldots,s_n)$ = payoff player i if pure strategies $s = (s_1,\ldots,s_n) \in S$ are played.
  - Nash equilibrium: Strategy profile $\sigma = (\sigma_1,\ldots,\sigma_n)$ such that for all players i, $u_i(\sigma_i,\sigma_{-i}) \geq u_i(\sigma_i',\sigma_{-i})$ for all σ_i'.
  - Brouwer theorem: Every continuous function $f: C \to C$ with C compact and convex has a fixed point $x \in C$, that is, $f(x) = x$.
  - Kakutani theorem: Every upper semi-continuous and convex-valued correspondence f on C (that is, for $x \in C$, $f(x) \subseteq C$), with C compact and convex, has a fixed point $x \in C$, that is, $x \in f(x)$.
  - Nash theorem: Every finite game $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ has a Nash equilibrium.
  - $n-$simplex $\Delta_n = \{(x_0,\ldots,x_n) \geq 0 \mid \sum_{i=1}^n x_i = 1\}$.
  - Correlated equilibrium: Probability distribution on outcome space $(p_s)_{s\in S}$, S be the set of all strategy profiles $S = (S_1,\ldots,S_n)$, that is self-enforcing, meaning that if the advice to player ℓ is to play pure strategy s_ℓ, then $\sum_{s_{-\ell}\in S_{-\ell}} p_{s_\ell s_{-\ell}} u_\ell(s_\ell, s_{-\ell}) \geq \sum_{s_{-\ell}\in S_{-\ell}} p_{s_\ell s_{-\ell}} u_\ell(s_\ell', s_{-\ell})$ for all $s_\ell' \in S_\ell$. Nash equilibria are (a special type of) correlated equilibria.

    Specifically, for two-player bimatrix games (A,B), a probability distribution (p_{ij}) on the outcome space is a correlated equilibrium if

$$\forall \text{ strategies } i = 1,\ldots,m : \sum_{j=1}^{n} p_{ij}(a_{ij} - a_{kj}) \geq 0 \text{ for all } k = 1,\ldots,m$$

$$\forall \text{ strategies } j = 1,\ldots,n : \sum_{i=1}^{m} p_{ij}(b_{ij} - b_{i\ell}) \geq 0 \text{ for all } \ell = 1,\ldots,n$$

- Extensive form games
  - Rooted directed tree with nodes v that correspond to either chance or decision nodes of the players. Several decision nodes (of one player) can form an information set h, meaning that the nodes $v \in h$ are indistinguishable for the player (hence, the possible actions at all $v \in h$ are identical).
  - Extensive form game has perfect recall if players recall their own past moves.
  - Extensive form game has perfect information if all information sets h are trivial, that is, consist of one node only.
  - Pure strategy s_i of player i: Precisely one action for each information set h of player i.
  - Behavioral strategy b_i of player i: For each information set h of player i, $b_i(h)$ is a probability distribution over the possible actions at h.
  - Nash equilibrium of an extensive form game: defined as Nash equilibrium of the corresponding strategic form game. (The pure strategies of player i in that strategic form game are formed by combination of one action of player i at all its informations sets h.)
  - Outcome equivalence: Two strategies of player i are outcome equivalent if, for each pure strategy profile s_{-i} of the other players, they generate the same distribution over the end nodes of the tree.
  - Subgame perfect equilibrium: Behavioral strategy that is a Nash equilibrium for each subgame induced by the game tree. (In particular, it is a Nash equilibrium for the whole game tree.)
  - Kuhn theorem: If extensive form game has perfect recall, any mixed strategy σ of the corresponding strategic form game has an outcome equivalent behavioral strategy b.

Cooperative Game Theory

- Cooperative games (N, v)
  - N = set of n players, $v : 2^N \to \mathbb{R}$ value function, $v(S)$ = worth of coalition S, $\mathbf{x} \in \mathbb{R}^n$ (usually) denotes a payoff vector, and for coalition $S \subseteq N$, $x(S) := \sum_{i \in S} x_i$.
  - Pre-imputation set = all efficient payoff vectors = $I^*(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}$.
  - Imputation set = all efficient and individually rational payoff vectors = $I(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \geq v(\{i\})\ \forall\ i \in N\}$.
  - Core $C(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S)\ \forall\ S \subseteq N\}$.
  - Payoff vector $\mathbf{z} \in I(N, v)$ is dominated via coalition S if there exists $\mathbf{y} \in I(N, v)$ such that $y_i > z_i$ for all $i \in S$ and $y(S) \leq v(S)$.
  - Domination core $DC(N, v) = \{\mathbf{x} \in I(N, v) \mid \mathbf{x} \text{ not dominated}\} = I(N, v) \backslash \bigcup_{\emptyset \neq S \subseteq N} D(S)$ where $D(S) := \{\mathbf{z} \in I(N, v) \mid \mathbf{z} \text{ dominated via } S \text{ by some } \mathbf{y} \in I(N, v)\}$.
- Special types of games
  - Game (N, v) is supper-additive if $v(S \cup T) \geq v(S) + v(T)\ \forall\ S \cap T = \emptyset$.
  - Game (N, v) is convex if $v : 2^N \to \mathbb{R}$ is supermodular, where supermodularity of v means $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\ \forall\ S, T$, or equivalently, for all $S \subseteq T \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.
  - Game (N, v) is balanced if for each balanced vector $\boldsymbol{\lambda}$, $\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$, where vector $\boldsymbol{\lambda} \in \mathbb{R}^{(2^n)} \geq 0$ is balanced if for all players i, $\sum_{S : i \in S} \lambda_S = 1$.
  - Bondareva-Shapley Theorem: $C(N, v) \neq \emptyset$ if and only if (N, v) balanced.
  - Simple games: $v(S) \in \{0, 1\}\ \forall\ S \subseteq N$ and $v(N) = 1$. Player i is veto player in a simple game if $(v(S) = 1 \Rightarrow i \in S)$.

- **The T-unanimity game**, for $T \subseteq N$, is the simple game (N, u_T) with

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise}. \end{cases}$$

- Solution values, concepts, etc.
  - Marginal (payoff) vector $\mathbf{m}^\sigma$: for given permutation σ of N, this is the payoffs when players enter a room in order σ and every player is handed out the marginal contribution, $m^\sigma_{\sigma(i)} = v(\sigma(1), \ldots, \sigma(i)) - v(\sigma(1), \ldots, \sigma(i-1))$.
  - Shapley value $\Phi(N, v) := \frac{1}{n!} \sum_\sigma \mathbf{m}^\sigma$. Also, for all $i \in N$, $\phi_i(N, v) = \frac{1}{n!} \sum_{S: i \notin S} |S|!(n - |S| - 1)!(v(S \cup \{i\}) - v(S))$.
  - Nucleolus = unique payoff vector $\mathbf{x}$ that lexicographically minimizes the vector of excesses $(e(S, \mathbf{x}))_{S \subseteq N}$, where excess of coalition S at $\mathbf{x}$, $e(S, \mathbf{x}) := v(S) - x(S)$. (In particular, it minimizes the maximal excess among all coalitions S.)
  - **Weber set** $\boldsymbol{W(N, v)} = \text{conv}\{\mathbf{m}^\sigma \mid \sigma \text{ permutation of } N\}$. $C(N, v) \subseteq W(N, v)$.
  - **Theorem** (Shapley, Ichiishi): $C(N, v) = W(N, v)$ if and only if (N, v) convex.
  - Harsanyi dividends: $\Delta(T) = v(T) - \sum_{S \subset T} \Delta(S)$, where $\Delta(\emptyset) = 0$.
  - Harsanyi theorem: For all $i \in N$, $\phi_i(N, v) = \sum_{T: i \in T} \frac{\Delta(T)}{|T|}$, with $\Phi(N, v)$ = Shapley value.
  - Null player i: $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$.
  - Symmetric players i, j: $v(S \cup \{i\}) = v(S \cup \{j\})$ for all S with $i, j \notin S$.
  - Value Ψ is efficient if $\Psi(N) = v(N)$, additive if $\Psi(v + w) = \Psi(v) + \Psi(w)$, symmetric if $\psi_i = \psi_j$ for symmetric players i, j, and has the null player property if $\psi_i = 0$ for null players i.
  - Shapley theorem: Shapley value = unique payoff vector that is efficient, additive, symmetric, and has the null player property.