Kenmerk: EWI2015/TW/DMMP/MU+JT

MSc Course Game Theory, 2015, code 191521800

Thursday, November 5, 2015, 08:45 - 11:45

Please motivate all your answers. Note: This exam comes with a cheat sheet that contains most of the basic definitions. (See the last three pages.) Other necessary definitions are given in the questions. You are also allowed to take your own cheat sheet (1 A4).

Noncooperative Game Theory

1. (6 points) Consider the (symmetric) bimatrix game given by

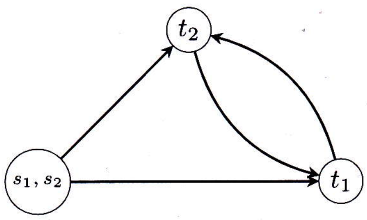
$$(A,B) = \begin{pmatrix} -5,-5 & 10,3 \\ 3,10 & 8,8 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game, and give a correlated equilibrium that is not a Nash equilibrium.

   (c) Explain in our own words (briefly!) how correlated equilibria extend the set of equilibria of a strategic form game. Can you also quantify that claim with the given example?

2. (4 points) Consider the network routing game depicted below. There are two players $i = 1, 2$, each of which has to select an (s_i, t_i) path. Each player can choose either the direct path, action d, or the indirect path, action i. All edges e have a cost that depends on the number of players choosing it, so that the cost function is $c_e(x) = x$, with x as the number of players choosing edge e.



   Suppose that player 1 chooses first, and then player 2. Model this game as an extensive form game, set up the payoff matrix of the corresponding strategic form game, and compute the pure Nash and subgame perfect equilibria. Briefly explain the outcome.

3. (4 points) Consider the following load balancing game. There are n tasks with processing times $p_j = 1$, $j = 1, \ldots, n$, that need to be distributed over $m < n$ machines. Machines have speeds $s_i \in (0, 1]$ (and we assume for simplicity that $1/s_i \in \mathbb{N}$). The latency experienced by any task equals the load of the machine it is processed on, divided by the speed of the machine. That is, if $N_i \subseteq N$ is the tasks on machine i, the latency of all tasks in N_i equals $1/s_i \sum_{j \in N_i} p_j = |N_i|/s_i$.

   Assume that tasks may selfishly select the machine to be processed on. Tasks are interested in a latency as small as possible. Like in network routing games, we are interested in minimising the total latency of all tasks.

   (a) Show that this game always has a pure strategy Nash equilibrium. (Hint: Define a suitable potential function.)

(b) Show that the price of anarchy[1] (for pure Nash equilibria) is > 1. (You can answer this question even if you have not been able to answer the previous one.)

Cooperative Game Theory

4. (4 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	1	2	5	14	15	11	20

   (a) Is the game essential? Is it superadditive? Is is it convex?

   (b) Compute the core $C(N, v)$ and the domination core $DC(N, v)$. (Are they equal?)

5. (3 points) Compute Weber Set W and Shapley value ϕ for the three player cooperative game of the previous question. Is $\phi \in C$? Is $W = C$?

6. (3 points) Show that an essential game has a nonempty core, $C \neq \emptyset$, for the case of two players. Also show that, for two players the Shapley value ϕ is always a core element, $\phi \in C$.

Stochastic Games

7. (6 points) Consider the following discounted stochastic game with infinite horizon and discount factor $\beta = \frac{1}{2}$:

state 1:

	-2 $(1,0)$	-3 $(0,1)$
	-3 $(1,0)$	-2 $(0,1)$

state 2:

1 $(0,1)$

   (a) Write down the set of equations that uniquely determines the value vector of this game.

   (b) Determine the value of this game, and optimal strategies for the players.

8. (6 points)

   (a) Consider the game The Big Match, with the average reward criterion:

state 1:

1 $(1,0,0)$	0 $(1,0,0)$
0 $(0,1,0)$	1 $(0,0,1)$

state 2:

0 $(0,1,0)$

state 3:

1 $(0,0,1)$

   The value vector is $\mathbf{v}_\alpha = (1/2, 0, 1)$. Show that the stationary strategy $\mathbf{g}_{\frac{1}{2}} = ((\frac{1}{2}, \frac{1}{2}), (1), (1))$ is an optimal strategy for player 2.

   (b) Let $(\mathbf{f}, \mathbf{g})$ be a pair of stationary strategies such that $P(\mathbf{f}, \mathbf{g})$ induces an irreducible Markov chain. Let the pair $(v, \mathbf{w})$ with $v \in \mathbb{R}$, $\mathbf{w} \in \mathbb{R}^N$, satisfy the equation

$$\mathbf{w} + v\mathbf{1}_N = \mathbf{r}(\mathbf{f}, \mathbf{g}) + P(\mathbf{f}, \mathbf{g})\mathbf{w}.$$

   Prove that $v_\alpha(s, \mathbf{f}, \mathbf{g}) = v$ for any s.

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[1] Recall that the price of anarchy (PoA) for an instance of a cost minimisation game is the ratio of the cost of a worst case Nash equilibrium over the cost of an optimal solution.

Basic definitions for MSc course on Game Theory

Noncooperative Game Theory

- Matrix games $A \in \mathbb{R}^{m \times n}$
  - Payoff row player $\mathbf{p}A\mathbf{q}$ with $\mathbf{p}$ = mixed strategy row player and $\mathbf{q}$ = mixed strategy column player. payoff column player $-\mathbf{p}A\mathbf{q}$.
  - Maximin strategy $\mathbf{p}$ for row player achieves maximum in $\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q}$. Minimax strategy $\mathbf{q}$ column player achieves minimum in $\min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}$.
  - von Neumann Theorem: $\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q} = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}$ for all (finite) A.
  - Entry (i, j) is saddlepoint if $a_{ij} \geq a_{kj}$ for all $k = 1, \ldots, m$ and $a_{ij} \leq a_{ik}$ for all $k = 1, \ldots, n$.
- Bimatrix Games (A, B) both $\in \mathbb{R}^{m \times n}$
  - Payoff row player $\mathbf{p}A\mathbf{q}$ with $\mathbf{p}$ = mixed strategy row player and $\mathbf{q}$ = mixed strategy column player, payoff column player $\mathbf{p}B\mathbf{q}$.
  - Carrier of strategy $\mathbf{p}$ for row player is $C(\mathbf{p}) = \{i \in \{1, \ldots, m\} : p_i > 0\}$, likewise for column player $C(\mathbf{q}) = \{j \in \{1, \ldots, n\} : q_j > 0\}$.
  - Nash equilibrium: strategy pair $(\mathbf{p}, \mathbf{q})$ such that $\mathbf{p}A\mathbf{q} \geq \mathbf{p}'A\mathbf{q}$ for all $\mathbf{p}'$ and $\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{q}'$ for all $\mathbf{q}'$.
  - Equilibrium principle: $(\mathbf{p}, \mathbf{q})$ is Nash equilibrium if and only if the pure strategies $C(\mathbf{p})$ are best replies to $\mathbf{q}$ and the pure strategies $C(\mathbf{q})$ are best replies to $\mathbf{p}$.
- Finite games $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$
  - N= set of n players, S_i = set of pure strategies of player i, $S = S_1 \times \cdots \times S_n$ = set of pure strategy profiles = set of possible outcomes, σ_i = mixed strategy of player i, $u_i(s) = u_i(s_1, \ldots, s_n)$ = payoff player i if pure strategies $s = (s_1, \ldots, s_n) \in S$ are played.
  - Nash equilibrium: Strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for all players i, $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all σ_i'.
  - Brouwer theorem: Every continuous function $f : C \to C$ with C compact and convex has a fixed point $x \in C$, that is, $f(x) = x$.
  - Kakutani theorem: Every upper semi-continuous and convex-valued correspondence f on C (that is, for $x \in C$, $f(x) \subseteq C$), with C compact and convex, has a fixed point $x \in C$, that is, $x \in f(x)$.
  - Nash theorem: Every finite game $G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N})$ has a Nash equilibrium.
  - n–simplex $\Delta_n = \{(x_0, \ldots, x_n) \geq 0 \mid \sum_{i=1}^{n} x_i = 1\}$.
  - Correlated equilibrium: Probability distribution on outcome space $(p_s)_{s \in S}$, S be the set of all strategy profiles $S = (S_1, \ldots, S_n)$, that is self-enforcing, meaning that if the advice to player ℓ is to play pure strategy s_ℓ, then $\sum_{s_{-\ell} \in S_{-\ell}} p_{s_\ell s_{-\ell}} u_\ell(s_\ell, s_{-\ell}) \geq \sum_{s_{-\ell} \in S_{-\ell}} p_{s_\ell s_{-\ell}} u_\ell(s_\ell', s_{-\ell})$ for all $s_\ell' \in S_\ell$. Nash equilibria are (a special type of) correlated equilibria.

    Specifically, for two-player bimatrix games (A, B), a probability distribution (p_{ij}) on the outcome space is a correlated equilibrium if

$$\forall \text{ strategies } i = 1, \ldots, m : \sum_{j=1}^{n} p_{ij}(a_{ij} - a_{kj}) \geq 0 \text{ for all } k = 1, \ldots, m$$

$$\forall \text{ strategies } j = 1, \ldots, n : \sum_{i=1}^{m} p_{ij}(b_{ij} - b_{i\ell}) \geq 0 \text{ for all } \ell = 1, \ldots, n$$

- Extensive form games
  - Rooted directed tree with nodes v that correspond to either chance or decision nodes of the players. Several decision nodes (of one player) can form an information set h, meaning that the nodes $v \in h$ are indistinguishable for the player (hence, the possible actions at all $v \in h$ are identical).
  - Extensive form game has perfect recall if players recall their own past moves.
  - Extensive form game has perfect information if all information sets h are trivial, that is, consist of one node only.
  - Pure strategy s_i of player i: Precisely one action for each information set h of player i.
  - Behavioral strategy b_i of player i: For each information set h of player i, $b_i(h)$ is a probability distribution over the possible actions at h.
  - Nash equilibrium of an extensive form game: defined as Nash equilibrium of the corresponding strategic form game. (The pure strategies of player i in that strategic form game are formed by combination of one action of player i at all its informations sets h.)
  - Outcome equivalence: Two strategies of player i are outcome equivalent if, for each pure strategy profile s_{-i} of the other players, they generate the same distribution over the end nodes of the tree.
  - Subgame perfect equilibrium: Behavioral strategy that is a Nash equilibrium for each subgame induced by the game tree. (In particular, it is a Nash equilibrium for the whole game tree.)
  - Kuhn theorem: If extensive form game has perfect recall, any mixed strategy σ of the corresponding strategic form game has an outcome equivalent behavioral strategy b.

Cooperative Game Theory

- Cooperative games (N, v)
  - N = set of n players, $v : 2^N \to \mathbb{R}$ value function, $v(S)$ = worth of coalition S, $\mathbf{x} \in \mathbb{R}^n$ (usually) denotes a payoff vector, and for coalition $S \subseteq N$, $x(S) := \sum_{i \in S} x_i$.
  - Game (N, v) is essential if $\sum_{i \in N} v(\{i\}) \leq v(N)$.
  - Pre-imputation set = all efficient payoff vectors = $I^*(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}$.
  - Imputation set = all efficient and individually rational payoff vectors = $I(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \geq v(\{i\})\ \forall\, i \in N\}$.
  - Core $C(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S)\ \forall\ S \subseteq N\}$.
  - Payoff vector $\mathbf{z} \in I(N, v)$ is dominated via coalition S if there exists $\mathbf{y} \in I(N, v)$ such that $y_i > z_i$ for all $i \in S$ and $y(S) \leq v(S)$.
  - Domination core $DC(N, v) = \{\mathbf{x} \in I(N, v) \mid \mathbf{x} \text{ not dominated}\} = I(N, v) \setminus \bigcup_{\emptyset \neq S \subseteq N} D(S)$ where $D(S) := \{\mathbf{z} \in I(N, v) \mid \mathbf{z} \text{ dominated via } S \text{ by some } \mathbf{y} \in I(N, v)\}$.
- Special types of games
  - Game (N, v) is supper-additive if $v(S \cup T) \geq v(S) + v(T)\ \forall\ S \cap T = \emptyset$.
  - Game (N, v) is convex if $v : 2^N \to \mathbb{R}$ is supermodular, where supermodularity of v means $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\ \forall\ S, T$, or equivalently, for all $S \subseteq T \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.
  - Game (N, v) is balanced if for each balanced vector $\boldsymbol{\lambda}$, $\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$, where vector $\boldsymbol{\lambda} \in \mathbb{R}^{(2^n)} \geq 0$ is balanced if for all players i, $\sum_{S : i \in S} \lambda_S = 1$.
  - Bondareva-Shapley Theorem: $C(N, v) \neq \emptyset$ if and only if (N, v) balanced.

- Simple games: $v(S) \in \{0,1\}\ \forall\ S \subseteq N$ and $v(N) = 1$. Player i is veto player in a simple game if $(v(S) = 1 \Rightarrow i \in S)$.
- The T-unanimity game, for $T \subseteq N$, is the simple game (N, u_T) with

$$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise}. \end{cases}$$

- Solution values, concepts, etc.
  - Marginal (payoff) vector $\mathbf{m}^\sigma$: for given permutation σ of N, this is the payoffs when players enter a room in order σ and every player is handed out the marginal contribution, $m^\sigma_{\sigma(i)} = v(\sigma(1), \ldots, \sigma(i)) - v(\sigma(1), \ldots, \sigma(i-1))$.
  - Shapley value $\Phi(N,v) := \frac{1}{n!}\sum_\sigma \mathbf{m}^\sigma$. Also, for all $i \in N$, $\phi_i(N,v) = \frac{1}{n!}\sum_{S:i \notin S} |S|!(n - |S| - 1)!(v(S \cup \{i\}) - v(S))$.
  - Nucleolus = unique payoff vector $\mathbf{x}$ that lexicographically minimizes the vector of excesses $(e(S,\mathbf{x}))_{S \subseteq N}$, where excess of coalition S at $\mathbf{x}$, $e(S,\mathbf{x}) := v(S) - x(S)$. (In particular, it minimizes the maximal excess among all coalitions S.)
  - Weber set $W(N,v) = \text{conv}\{\mathbf{m}^\sigma \mid \sigma \text{ permutation of } N\}$. $C(N,v) \subseteq W(N,v)$.
  - Theorem (Shapley, Ichiishi): $C(N,v) = W(N,v)$ if and only if (N,v) convex.
  - Harsanyi dividends: $\Delta(T) = v(T) - \sum_{S \subset T} \Delta(S)$, where $\Delta(\emptyset) = 0$.
  - Harsanyi theorem: For all $i \in N$, $\phi_i(N,v) = \sum_{T:i \in T} \frac{\Delta(T)}{|T|}$, with $\Phi(N,v)$ = Shapley value.
  - Null player i: $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$.
  - Symmetric players i,j: $v(S \cup \{i\}) = v(S \cup \{j\})$ for all S with $i,j \notin S$.
  - Value Ψ is efficient if $\Psi(N) = v(N)$, additive if $\Psi(v+w) = \Psi(v) + \Psi(w)$, symmetric if $\psi_i = \psi_j$ for symmetric players i,j, and has the null player property if $\psi_i = 0$ for null players i.
  - Shapley theorem: Shapley value = unique payoff vector that is efficient, additive, symmetric, and has the null player property.