Exam Game Theory (191521800)
University of Twente
November 8, 2018, 8:45-11:45h

This exam has 8 exercises.
Motivate all your answers! You may not use any electronic device.
This exam comes with a cheat sheet that contains most of the basic definitions. (See the last pages.) Other necessary definitions are given in the questions. You are also allowed to bring your own cheat sheet (1 A4, one-sided).

Noncooperative Game Theory

1. (6 points) Consider the (symmetric) bimatrix game given by

$$(A, B) = \begin{pmatrix} -2, 1 & 14, 9 \\ 6, 16 & 12, 14 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game, and give a correlated equilibrium that is not a Nash equilibrium.

   (c) Give the correlated equilibrium that maximizes the total utility of the players.

2. (6 points) Consider the following *cost sharing game*, played on an undirected graph $G = (V, E)$ with a destination node $t \in V$. Each node $v \in V \setminus \{t\}$ is a player that seeks to build a connection to t. Hence, set of the strategies of a player v is the set of paths from v to t.

   Each edge $e \in E$ has a cost of $c_e > 0$ that is shared among all its users. For a strategy profile s, let $n_e(s) = |\{v \mid e \in s_v\}|$ denotes the number of players that choose a path containing edge e in s. The cost of a player $v \in V \setminus \{t\}$ in s is $c_v(s) := \sum_{e \in s_v} \frac{c_e}{n_e(s)}$.

   (a) Show that this game always has a pure strategy Nash equilibrium.
   (Hint: Define a suitable potential function $p(s)$ that strictly decreases along with any improving move of a player.)

   (b) Suppose we are interested in finding a strategy profile that minimizes the total cost. Show that the best pure strategy Nash equilibrium of the above game costs at most H_n times the minimal cost $\min_{s \in S} \sum_{v \in V \setminus \{t\}} c_v(s)$ – i.e., the price of stability is at most the n-th harmonic number $H_n = \sum_{i=1}^{n} \frac{1}{i}$ where n denote the number of players.

Cooperative Game Theory

3. (4 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	3	5	6	9	8	15

   (a) Is the game essential? Is it superadditive? Is it convex?

   (b) Compute the Weber set $W(N, v)$ for this game, and express it as convex hull of its extreme points. What is the core $C(N, v)$ for this game?

   (c) Is the Shapley value of this game in the Core?

4. (4 points) Show that, for the case of two players, an essential game has a nonempty core, $C \neq \emptyset$. Also show that, for two players, the Shapley value ϕ is always a core element, $\phi \in C$.

5. (4 points) Consider the following solution value ψ defined for a cooperative game (N, v) with $|N| = n$. For all $i \in N$, let i's payoff be

$$\psi_i(N, v) := \frac{v(\{i\})}{\sum_{j=1}^{n} v(\{j\})} v(N) \,.$$

   Show that ψ is efficient and symmetric and fulfills the null player property, but may fail to be additive (in general).

Stochastic Game Theory

6. (5 points) Consider the infinitely repeated discounted game $G^\infty(\delta)$ with discount factor $\delta \in (0, 1)$ and

$$G = \begin{array}{c} \\ T \\ B \end{array} \begin{array}{c} \begin{array}{cc} T & \quad B \end{array} \\ \begin{pmatrix} 15, 13 & 9, 14 \\ 20, 8 & 10, 10 \end{pmatrix} \end{array}$$

   (a) Consider the strategy $Tr(T)$: at $t = 0$ and every time t such that in the past only (T, T) has occurred in the stage game, play T. Otherwise, play B. For which values of δ is $(Tr(T), Tr(T))$ a subgame perfect Nash equilibrium?

   (b) According to the Folk Theorem for subgame perfect equilibrium, which payoffs can be reached as limit average payoffs in a subgame perfect Nash equilibrium of $G^\infty(\delta)$? (Hint: (B, B) is the unique Nash equilibrium in G.)

7. (4 points) Consider a discounted stochastic game with value vector $v_\beta = (v_\beta(1), \ldots, v_\beta(N))$. Assume both players possess optimal stationary strategies. Prove that $v_\beta(s) = \text{val}[R_\beta(s, v_\beta)]$, where $R_\beta(s, x)$ is the $m^1(s) \times m^2(s)$ matrix game with entry $(1 - \beta) r(s, a^1, a^2) + \beta \sum_{s' \in S} p(s'|s, a^1, a^2) x(s')$ in cell (a^1, a^2).

8. (3 points) The stochastic game

0 $(0, 1)$	1 $(\frac{1}{2}, \frac{1}{2})$
1 $(\frac{1}{2}, \frac{1}{2})$	-2 $(0, 1)$

state 1

4 $(\frac{1}{2}, \frac{1}{2})$

state 2

is an irreducible game. The players optimize their average rewards. The players 1 and 2 use stationary strategies $\mathbf{f} = ((1, 0), (1))$ and $\mathbf{g} = ((\frac{1}{2}, \frac{1}{2}), (1))$ respectively. Determine the value vector $\mathbf{v}_\alpha(\mathbf{f}, \mathbf{g})$.

Total: 36 + 4 = 40 points