Exam Game Theory (191521800)

University of Twente

November 7, 2019, 8:45-11:45h

This exam has 7 exercises.

Motivate all your answers! **You may not use any electronic device.**

This exam comes with a cheat sheet that contains most of the basic definitions. (See the last pages.) Other necessary definitions are given in the questions. You are also allowed to bring your own cheat sheet (1 A4, one-sided).

Noncooperative Game Theory

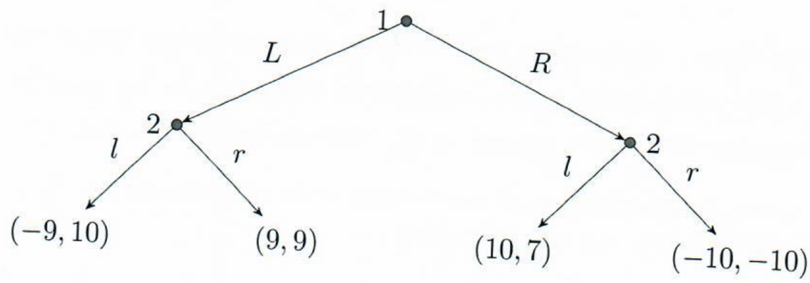
1. (5 points) Consider the bimatrix game given by

$$(A, B) = \begin{pmatrix} 12, 14 & 7, 16 \\ 14, 9 & -1, 1 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game, and give a correlated equilibrium that is not a Nash equilibrium.

2. (5 points) Consider the following extensive form game (with perfect information and perfect recall).



   (a) Give the strategic form (bimatrix) representation of this extensive form game.

   (b) Compute all subgame perfect equilibria.

   (c) Give a Nash equilibrium that is not a subgame perfect equilibrium

3. (6 points) A *network cost sharing game* is an atomic network routing game with latency (or cost) functions on the arcs of the form $\ell_e(x) = \frac{1}{x}$.

   (a) Show that this game always has a pure Nash equilibrium.
   (Hint: Use a suitable potential function and show it strictly decreases with any improving move of a player.)

   (b) Show the price of stability is at most H_n, where H_n is the n-th harmonic number, i.e., $H_n = \sum_{i=1}^{n} \frac{1}{i}$.
   (Hint: Argue about a Nash equilibrium that is reached after improving moves starting from the social optimum P^*.)

Cooperative Game Theory

4. (5 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	3	4	5	6	8	10

   (a) Is the game essential? Is it superadditive? Is it convex?

   (b) Compute the core $C(N, v)$ for this game, and express it as convex hull of its extreme points.

   (c) What is the maximal value of $v(\{2\})$ such that the core still is nonempty?

5. (5 points) We call a cooperative game (N, v) *additive* if for every $S, T \subseteq N$ with $S \cap T = \emptyset$, we have $v(S \cup T) = v(T) + v(S)$.

   (a) Show that the Shapley value of an additive game is $\Phi_i(N, v) = v(\{i\})$.

   (b) Give the nucleolus of an arbitrary additive game (N, v).

Stochastic Game Theory

6. (7 points) Consider the following transient stochastic game.

1 $(\frac{1}{3}, \frac{2}{3})$	2 $(0, 1)$
4 $(0, 1)$	3 $(\frac{1}{2}, \frac{1}{2})$

state 1

1   $(0, \frac{1}{2})$

state 2

   (a) Write down the set of equations that uniquely determines the value vector of this game.

   (b) Determine the value of this game and optimal strategies for the players.

   (c) Explain whether or not this game is a discounted stochastic game.

7. (3 points) Consider a discounted stochastic game with value vector v_β. Let f^* be optimal stationary strategy for player 1. Then one can show that

$$\min_g \left\{ (1-\beta) r(s, f^*, g) + \beta \sum_{s' \in S} p(s'|s, f^*, g) v_\beta(s') \right\} \geq v_\beta(s).$$

Analogously,

$$\max_f \left\{ (1-\beta) r(s, f, g^*) + \beta \sum_{s' \in S} p(s'|s, f, g^*) v_\beta(s') \right\} \leq v_\beta(s)$$

for optimal stationary strategy g^* for player 2. Show how this implies that $\text{val}[R_\beta(s, v_\beta)] = v_\beta(s)$. ($R_\beta(s, x)$ is the $m^1(s) \times m^2(s)$ matrix game with $(1-\beta) r(s, a^1, a^2) + \beta \sum_{s' \in S} p(s'|s, a^1, a^2) x(s')$ in cell (a^1, a^2).) □

Total: 36 + 4 = 40 points