Exam Game Theory (191521800)
University of Twente
November 5, 2020, 9:00-12:00h

This exam has 7 exercises.
Motivate all your answers! **You may not use any electronic device.**
This exam comes with a cheat sheet that contains most of the basic definitions. (See the last pages.) Other necessary definitions are given in the questions. You are also allowed to bring your own cheat sheet (1 A4, one-sided).

Noncooperative Game Theory

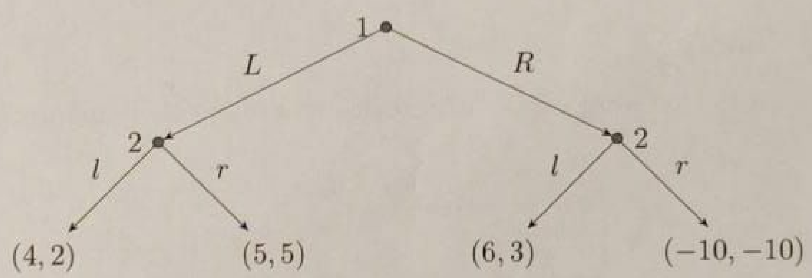
1. (4 points) Consider the bimatrix game given by

$$(A,B) = \begin{array}{cc} \begin{matrix} q & 1-q \end{matrix} & \\ \begin{pmatrix} 12,14 & 7,16 \\ 14,9 & -1,1 \end{pmatrix} & \begin{matrix} p \\ 1-p \end{matrix} \end{array}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game, and give a correlated equilibrium that is not a Nash equilibrium.

2. (4 points) Consider the following extensive form game (with perfect information and perfect recall).



   (a) Give the strategic form (bimatrix) representation of this extensive form game.

   (b) Compute all **pure** Nash equilibria.

   (c) Compute all subgame perfect equilibria.

3. (5 points) A *selfish scheduling game* is a finite strategic game with n players and m machines. Every player chooses a machine. So the set of pure strategies for a player $i \in \{1, \ldots, n\}$ is $S_i = \{1, \ldots, m\}$. For a strategy profile $s = (s_1, \ldots, s_n)$ and a machine j, we denote by $n_j(s)$ the number of players that choose machine j in s.

   The payoff of a player i is defined as $u_i(s) = -n_{s_i}(s)$. That is, her cost corresponds to the number of players on her chosen machine.

   (a) Show that this game always has a pure Nash equilibrium.
   (Hint: Use a suitable potential function and show it strictly decreases with any improving move of a player.)

   (b) Show the price of stability is 1.

Cooperative Game Theory

4. (5 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	3	4	5	6	8	10

   (a) Is it superadditive? Is it convex?

   (b) Compute the core $C(N, v)$ for this game, and express it as convex hull of its extreme points.

5. (2 points) Compute the Shapley value ϕ for the three player cooperative game of the previous question. Is $\phi \in C$?

6. (6 points) Let (N, v) be an arbitrary **convex** cooperative game.
   We define the *superman game* of (N, v) by adding a player called s (superman) as (N', v') with $N' = N \cup \{s\}$ and

$$v'(S) = \begin{cases} v(S) + 100 & s \in S \\ v(S) & else \end{cases}$$

   (a) What is the Shapley value $\Phi_s(N', v')$ of player s in (N', v')?

   (b) Is (N', v') is a convex game?

   (c) Prove the following statement:
   If $(x_1, \ldots, x_n) \in C(N, v)$ then $(x_1, \ldots, x_n, 100) \in C(N', v')$.

   (Hint: Think about the marginal vectors in both games.)

Stochastic Game Theory

6. (7 points) Consider the following irreducible stochastic game with an infinite horizon and the average-reward criterion

	1 $(\frac{1}{2}, \frac{1}{2})$	5 $(0, 1)$
	3 $(0, 1)$	2 $(\frac{1}{4}, \frac{3}{4})$

state 1

-1 $(\frac{1}{2}, \frac{1}{2})$

state 2

and decision rules $\mathbf{f} = ((1, 0), (1))$ and $\mathbf{g} = ((0, 1), (1))$ for players 1 and 2, respectively.

   (a) Calculate $v_\alpha(s, \mathbf{f}, \mathbf{g})$ for all states s.

   (b) Solve the vector equation $w + v1_N = r(\mathbf{f}, \mathbf{g}) + P(\mathbf{f}, \mathbf{g})w$ for $v \in \mathbb{R}$ and $w \in \mathbb{R}^2$ and interpret the solutions v and w.

   (c) Is the strategy pair $(\mathbf{f}^\infty, \mathbf{g}^\infty)$ average optimal?

   (d) Give an example of a history-dependent strategy for player 2 in this game.

7. (3 points)

   (a) Mention two (nontrivial) differences between infinite-horizon zero-sum stochastic games with the discounted-reward criterion and the average-reward criterion.

   (b) Consider a zero-sum β-discounted stochastic game Γ_β. Prove that if (π^1_*, π^2_*) is an equilibrium point in Γ_β then π^1_* is optimal for player 1, π^2_* is optimal for player 2, and $\mathbf{v}_\beta(\pi^1_*, \pi^2_*) = \mathbf{v}_\beta$.

Total: 36 + 4 = 40 points