Exam Game Theory (191521800)
University of Twente
November 10, 2021, 8:45-11:45h

This exam has 8 exercises.
Motivate all your answers! **You may not use any electronic device.**
This exam comes with a cheat sheet that contains most of the basic definitions. (See the last pages.) Other necessary definitions are given in the questions. You are also allowed to bring your own cheat sheet (1 A4, one-sided).

Noncooperative Game Theory

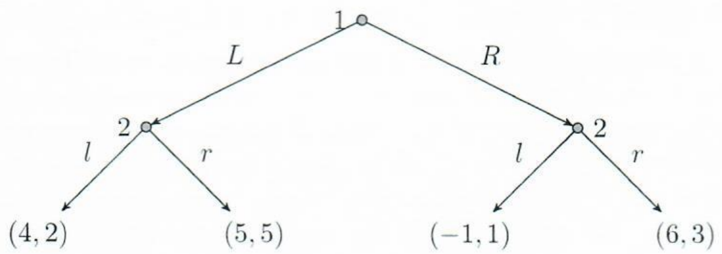
1. (2+2 points) Consider the bimatrix game given by

$$(A,B) = \begin{pmatrix} 12,14 & 7,16 & 8,15 \\ 14,5 & -1,1 & 9,8 \end{pmatrix}$$

   (a) Compute all Nash equilibria of this game.

   (b) Write down all conditions that define the correlated equilibria of this game.

2. (1+2+1 points) Consider the following extensive form game (with perfect information and perfect recall).



   (a) Give the strategic form (bimatrix) representation of this extensive form game.

   (b) Compute all pure Nash equilibria and also give the corresponding behavioural strategies.

   (c) Compute all subgame perfect equilibria.

3. (4 points) Prove that for every matrix game $A^{m\times n}$ it is true that $v_1(A) \le v_2(A)$. You may not use the von Neumann Theorem.

Cooperative Game Theory

4. (1+3+3 points) Consider the following three player cooperative game (N, v).

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v(S)$	2	3	4	5	6	8	10

   (a) Is it superadditive? Is it convex?

   (b) Compute the core $C(N, v)$ for this game, and express it as convex hull of its extreme points.

   (c) Compute the Shapley value. Is the Shapley value of this game in the core?

5. (3 points) We call a game (N, v) strictly convex if $v(S) + v(T) < v(S \cup T) + v(S \cap T)$ for all $S, T \subseteq N$. Show that in a strictly convex game all marginal vectors are different.

6. (2 points) Show that the Shapley value of every two player game (N, v) is in the core if $C(N, v) \neq \emptyset$.

Stochastic Game Theory

7. (1+3+2+1 points) Consider the stochastic game situation below in which the players optimize their average rewards.

1 ; $(\frac{1}{2}, \frac{1}{2})$	2 ; $(0, 1)$
3 ; $(0, 1)$	1 ; $(\frac{1}{2}, \frac{1}{2})$

$s = 1$

-1 ; $(\frac{1}{4}, \frac{3}{4})$

$s = 2$

   (a) Why is this game irreducible?

   (b) The players 1 and 2 use stationary strategies $\mathbf{f}^\infty$ with $\mathbf{f} = ((\frac{1}{2}, \frac{1}{2}), 1)$ and $\mathbf{g}^\infty$ with $\mathbf{g} = ((\frac{2}{5}, \frac{3}{5}), 1)$, respectively. Determine the value vector $v_\alpha(\mathbf{f}, \mathbf{g})$.

   (c) Find another stationary strategy that player 1 would prefer over $\mathbf{f}^\infty$. Assume player 2 remains using $\mathbf{g}^\infty$.

   (d) Give an example of a Markov strategy for player 1 in this game.

8. (2+3 points)

   (a) Let (π^1_*, π^2_*) and (π^1_{**}, π^2_{**}) be two equilibrium points of a zero-sum discounted stochastic game. Prove that $v^k_\beta(\pi^1_*, \pi^2_*) = v^k_\beta(\pi^1_{**}, \pi^2_{**})$ for player $k = 1, 2$.

   (b) In a tv show, the two players play a game during three rounds. In the first round, player 2 places a gold or silver trophy, worth 2000 and 1000 Euros respectively, behind a curtain. Player 1 should guess which trophy is behind the curtain. If she guessed correctly which trophy is behind the curtain, she receives the value of the trophy in Euros from player 2 and the game continues to round 2 where player 2 again puts a gold or silver trophy behind the curtain. And so on. If player 1 guessed wrong, she gets nothing in this round and the remaining ones. After three rounds, the game ends. Player 1 aims to maximise her reward while player 2 wants to minimise his payout to player 1. Model this situation as a discounted stochastic game with finite horizon and $\beta = 1$.

Total: 36 + 4 = 40 points

Basic definitions for MSc course on Game Theory

Noncooperative Game Theory

- Matrix games $A \in \mathbb{R}^{m\times n}$, $\mathbf{a}^j$ = column j, $\mathbf{a}_i$ = row i of a matrix A
  - Payoff row player (player 1) is $\mathbf{p}A\mathbf{q}$ with $\mathbf{p}$ = mixed strategy row player and $\mathbf{q}$ = mixed strategy column player. Payoff column player (player 2) is $-\mathbf{p}A\mathbf{q}$.
  - Maximin strategy $\mathbf{p}$ for row player achieves maximum in $\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q}$. Minimax strategy $\mathbf{q}$ column player achieves minimum in $\min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}$.
  - von Neumann Theorem: $\max_{\mathbf{p}} \min_{\mathbf{q}} \mathbf{p}A\mathbf{q} = \min_{\mathbf{q}} \max_{\mathbf{p}} \mathbf{p}A\mathbf{q}$ for all (finite) A.
  - Entry (i, j) is saddlepoint if $a_{ij} \geq a_{kj}$ for all $k = 1, \ldots, m$ and $a_{ij} \leq a_{ik}$ for all $k = 1, \ldots, n$. In words: a_{ij} is maximal in column $\mathbf{a}^j$ and minimal in row $\mathbf{a}_i$. a_{ij} = saddlepoint $\Leftrightarrow$ strategies $\mathbf{e}^i$ for player 1 and $\mathbf{e}_j$ for player 2 are a (pure) Nash equilibrium
- Bimatrix Games (A, B) both $\in \mathbb{R}^{m\times n}$
  - Payoff row player $\mathbf{p}A\mathbf{q}$ with $\mathbf{p}$ = mixed strategy row player and $\mathbf{q}$ = mixed strategy column player, payoff column player $\mathbf{p}B\mathbf{q}$.
  - Carrier of strategy $\mathbf{p}$ for row player is $C(\mathbf{p}) = \{i \in \{1, \ldots, m\} : p_i > 0\}$, likewise for column player $C(\mathbf{q}) = \{j \in \{1, \ldots, n\} : q_j > 0\}$.
  - Nash equilibrium: strategy pair $(\mathbf{p}, \mathbf{q})$ such that $\mathbf{p}A\mathbf{q} \geq \mathbf{p}'A\mathbf{q}$ for all $\mathbf{p}'$ and $\mathbf{p}B\mathbf{q} \geq \mathbf{p}B\mathbf{q}'$ for all $\mathbf{q}'$.
  - Equilibrium principle: $(\mathbf{p}, \mathbf{q})$ is Nash equilibrium if and only if the pure strategies $C(\mathbf{p})$ are all best replies to $\mathbf{q}$ and the pure strategies $C(\mathbf{q})$ are all best replies to $\mathbf{p}$. In words: both player only play those strategies (with positive probability) that give maximal payoff. This is equivalent with: $\mathbf{e}^j A\mathbf{q} \geq \mathbf{e}^k A\mathbf{q}$ for all $j \in C(\mathbf{p})$ and all $k = 1, \ldots, m$ and $\mathbf{p}B\mathbf{e}_i \geq \mathbf{p}B\mathbf{e}_k$ for all $i \in C(\mathbf{q})$ and all $k = 1, \ldots, n$.
- Finite games $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$
  - N= set of n players, S_i = set of pure strategies of player i, $S = S_1 \times \cdots \times S_n$ = set of pure strategy profiles = set of possible outcomes, σ_i = mixed strategy of player i, $u_i(s) = u_i(s_1, \ldots, s_n)$ = payoff player i if pure strategies $s = (s_1, \ldots, s_n) \in S$ are played.
  - Nash equilibrium: Strategy profile $\sigma = (\sigma_1, \ldots, \sigma_n)$ such that for all players i, $u_i(\sigma_i, \sigma_{-i}) \geq u_i(\sigma_i', \sigma_{-i})$ for all σ_i'.
  - Brouwer theorem: Every continuous function $f : C \to C$ with C compact and convex has a fixed point $x \in C$, that is, $f(x) = x$.
  - Kakutani theorem: Every upper semi-continuous and convex-valued correspondence f on C (that is, for $x \in C$, $f(x) \subseteq C$), with C compact and convex, has a fixed point $x \in C$, that is, $x \in f(x)$.
  - Nash theorem: Every finite game $G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$ has a Nash equilibrium.

- n−simplex $\Delta_n = \{(x_0, \ldots, x_n) \geq 0 \mid \sum_{i=1}^{n} x_i = 1\}$.
- Correlated equilibrium: Probability distribution on outcome space $(p_s)_{s \in S}$, S being the set of all strategy profiles $S = (S_1, \ldots, S_n)$, that is self-enforcing, meaning that if the advice to player ℓ is to play pure strategy s_ℓ, then $\sum_{s_{-\ell} \in S_{-\ell}} p_{s_\ell s_{-\ell}} u_\ell(s_\ell, s_{-\ell}) \geq \sum_{s_{-\ell} \in S_{-\ell}} p_{s_\ell s_{-\ell}} u_\ell(s'_\ell, s_{-\ell})$ for all $s'_\ell \in S_\ell$. Nash equilibria are (a special type of) correlated equilibria.

  Specifically, for two-player bimatrix games (A, B), a probability distribution $P = (p_{ij})_{i=1,\ldots,m, j=1,\ldots,m}$ on the outcome space is a correlated equilibrium if

  $$\forall \text{ strategies } i = 1, \ldots, m : \sum_{j=1}^{n} p_{ij}(a_{ij} - a_{kj}) \geq 0 \text{ for all } k = 1, \ldots, m$$

  $$\forall \text{ strategies } j = 1, \ldots, n : \sum_{i=1}^{m} p_{ij}(b_{ij} - b_{i\ell}) \geq 0 \text{ for all } \ell = 1, \ldots, n$$

  $P = (p_{ij})$ is a Nash equilibrium $\Leftrightarrow$ P has rank 1 and is a correlated equilibrium.

- Extensive form games
  - Rooted directed tree with nodes v that correspond to either chance or decision nodes of the players. Several decision nodes (of one player) can form an information set h, meaning that the nodes $v \in h$ are indistinguishable for the player (hence, the chosen actions at all $v \in h$ must be identical).
  - Extensive form game has perfect recall if players recall their own past moves.
  - Extensive form game has perfect information if all information sets h are trivial, that is, consist of one node only.
  - Pure strategy s_i of player i: Precisely one action for each information set h of player i.
  - Behavioral strategy b_i of player i: For each information set h of player i, $b_i(h)$ is a probability distribution over the possible actions at h.
  - Nash equilibrium of an extensive form game: defined as Nash equilibrium of the corresponding strategic form game. (The pure strategies of player i in that strategic form game are formed by combination of one action of player i at all its informations sets h.)
  - Outcome equivalence: Two strategies of player i are outcome equivalent if, for each pure strategy profile s_{-i} of the other players, they generate the same distribution over the end nodes of the tree.
  - Subgame perfect equilibrium: Behavioral strategy that is a Nash equilibrium for each subgame induced by the game tree. (In particular, it is a Nash equilibrium for the whole game tree.)
  - Kuhn theorem: If extensive form game has perfect recall, any mixed strategy σ of the corresponding strategic form game has an outcome equivalent behavioral strategy b.

Cooperative Game Theory

- Cooperative games (N, v)
  - N = set of n players, $v : 2^N \to \mathbb{R}$ value function, $v(S)$ = worth of coalition S, $\mathbf{x} \in \mathbb{R}^n$ (usually) denotes a payoff vector, and for coalition $S \subseteq N$, $x(S) := \sum_{i \in S} x_i$.
  - Game (N, v) is essential if $\sum_{i \in N} v(\{i\}) \leq v(N)$.
  - Pre-imputation set = all efficient payoff vectors = $I^*(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N)\}$.
  - Imputation set = all efficient and individually rational payoff vectors = $I(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N),\ x_i \geq v(\{i\})\ \forall\ i \in N\}$.
  - Core $C(N, v) = \{\mathbf{x} \in \mathbb{R}^n \mid x(N) = v(N),\ x(S) \geq v(S)\ \forall\ S \subseteq N\}$.
  - Payoff vector $\mathbf{z} \in I(N, v)$ is dominated via coalition S if there exists $\mathbf{y} \in I(N, v)$ such that $y_i > z_i$ for all $i \in S$ and $y(S) \leq v(S)$.
  - Domination core $DC(N, v) = \{\mathbf{x} \in I(N, v) \mid \mathbf{x} \text{ not dominated}\} = I(N, v) \setminus \bigcup_{\emptyset \neq S \subseteq N} D(S)$ where $D(S) := \{\mathbf{z} \in I(N, v) \mid \mathbf{z} \text{ dominated via } S \text{ by some } \mathbf{y} \in I(N, v)\}$.
- Special types of games
  - Game (N, v) is supper-additive if $v(S \cup T) \geq v(S) + v(T)\ \forall\ S \cap T = \emptyset$.
  - Game (N, v) is convex if $v : 2^N \to \mathbb{R}$ is supermodular, where supermodularity of v means $v(S \cup T) + v(S \cap T) \geq v(S) + v(T)\ \forall\ S, T$, or equivalently, for all $S \subseteq T \subseteq N \setminus \{i\}$, $v(S \cup \{i\}) - v(S) \leq v(T \cup \{i\}) - v(T)$.
  - Note that a convex game (N, v) is also super-additive, but not vice versa.
  - Game (N, v) is balanced if for each balanced vector $\boldsymbol{\lambda}$, $\sum_{S \subseteq N} \lambda_S v(S) \leq v(N)$, where vector $\boldsymbol{\lambda} \in \mathbb{R}^{(2^n)} \geq 0$ is balanced if for all players i, $\sum_{S : i \in S} \lambda_S = 1$.
  - Bondareva-Shapley Theorem: $C(N, v) \neq \emptyset$ if and only if (N, v) balanced.
  - Simple games: $v(S) \in \{0, 1\}\ \forall\ S \subseteq N$ and $v(N) = 1$. Player i is veto player in a simple game if $(v(S) = 1 \Rightarrow i \in S)$.
  - The T-unanimity game, for $T \subseteq N$, is the simple game (N, u_T) with

    $$u_T(S) = \begin{cases} 1 & \text{if } T \subseteq S \\ 0 & \text{otherwise}. \end{cases}$$

    The set of all T-unanimity games u_T, $\emptyset \neq T \subseteq N$, forms a basis for the $2^{|N|} - 1$ dimensional vector space of all cooperative games with $|N|$ players.
- Solution values, concepts, etc.
  - Marginal (payoff) vector $\mathbf{m}^\sigma$: for given permutation σ of N, this is the payoffs when players enter a room in order σ and every player is handed out the marginal contribution, $m^\sigma_{\sigma(i)} = v(\sigma(1), \ldots, \sigma(i)) - v(\sigma(1), \ldots, \sigma(i-1))$.
  - Shapley value $\Phi(N, v) := \frac{1}{n!} \sum_\sigma \mathbf{m}^\sigma$. Also, for all $i \in N$, $\phi_i(N, v) = \frac{1}{n!} \sum_{S : i \notin S} |S|!(n - |S| - 1)!(v(S \cup \{i\}) - v(S))$.

- Nucleolus = unique payoff vector $\mathbf{x}$ that lexicographically minimizes the vector of excesses $(e(S, \mathbf{x}))_{S \subseteq N}$, where excess of coalition S at $\mathbf{x}$, $e(S, \mathbf{x}) := v(S) - x(S)$. (In particular, it minimizes the maximal excess among all coalitions S.)
- Weber set $W(N, v) = \text{conv}\{\mathbf{m}^\sigma \mid \sigma \text{ permutation of } N\}$. $C(N, v) \subseteq W(N, v)$.
- Theorem (Shapley, Ichiishi): $C(N, v) = W(N, v)$ if and only if (N, v) convex.
- Harsanyi dividends: $\Delta(T) = v(T) - \sum_{S \subset T} \Delta(S)$, where $\Delta(\emptyset) = 0$.
- Harsanyi theorem: For all $i \in N$, $\phi_i(N, v) = \sum_{T: i \in T} \frac{\Delta(T)}{|T|}$, with $\Phi(N, v) =$ Shapley value.
- Null player i: $v(S \cup \{i\}) - v(S) = 0$ for all $S \subseteq N$.
- Symmetric players i, j: $v(S \cup \{i\}) = v(S \cup \{j\})$ for all S with $i, j \notin S$.
- Value Ψ is efficient if $\Psi(N) = v(N)$, additive if $\Psi(v+w) = \Psi(v) + \Psi(w)$, symmetric if $\psi_i = \psi_j$ for symmetric players i, j, and has the null player property if $\psi_i = 0$ for null players i.
- Shapley theorem: Shapley value = unique payoff vector that is efficient, additive, symmetric, and has the null player property.

Price of Anarchy for (Atomic) Network Routing

- An (atomic) network routing game is a directed graph $G = (V, E)$, with latency (or cost) functions $\ell_e(x)$ for each arc $e \in E$, n players $i = 1, \ldots, n$ and origin-destination pairs (s_i, t_i) with $s_i, t_i \in V$ for all i. The strategy space of player i consists of all (s_i, t_i)-paths $\mathcal{P}_i$ in G.
- The (atomic) network routing game is symmetric if $(s_i, t_i) = (s_k, t_k)$ for all players i, k.
- A strategy profile is $(P_1, \ldots, P_n)$ with $P_i \in \mathcal{P}_i$
- The latency (or cost) of player i in profile P equals $\sum_{e \in P_i} \ell_e(n_e(P))$, with $n_e(P) =$ total number of players k with $e \in P_k$, or $n_e(P) = \sum_{i=1}^{n} |\{e\} \cap P_i|$.
- The total latency (or "social cost") is $\ell(P) := \sum_{i=1}^{n} \sum_{e \in P_i} \ell_e(n_e(P)) = \sum_{e \in E} n_e(P) \cdot \ell_e(n_e(P))$.
- The price of anarchy (for the total latency) is $\max_{P \in NE} \ell(P)/\ell(OPT)$ where $OPT =$ strategy profile minimizing total latency $\ell(\cdot)$ and $NE =$ set of all Nash equilibrium strategy profiles.
- The price of stability (for the total latency) is $\min_{P \in NE} \ell(P)/\ell(OPT)$ where $OPT =$ strategy profile minimizing total latency $\ell(\cdot)$ and $NE =$ set of all Nash equilibrium strategy profiles.
- A potential function for strategy profile P is $\sum_{e \in E} [\ell_e(1) + \cdots + \ell_e(n_e(P))]$ (also known as Rosenthal potential)